Achievable Rate Region under Joint Distributed Beamforming and Power Allocation for Two-Way Relay Networks

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Abstract—We obtain the achievable beamforming rate region for a two-way cooperative network consisting of two transceivers and multiple relays, all single-antenna nodes. Assuming that the relay beamforming weights as well as the transceiver transmit powers are the design parameters, this region is characterized under a total (network) transmit power consumption constraint. Using the shape of the rate region, we then use a sum-rate maximization approach to obtain the jointly optimal relay beamforming weights and transceiver transmit powers. Interestingly, we show that the sum-rate maximization approach yields the same solution as the max-min fair design approach does. Using this relationship, we further present a semi-closed-form solution to the underlying distributed beamforming problem. We then prove that the transmit power of any of the two transceivers can be obtained as the solution to a one-dimensional optimization problem using a simple bisection method which enjoys a low computational complexity. Furthermore, we extend these results to obtain the relay beamforming weights and transceiver transmit powers corresponding to any point on the boundary of the rate region, through a weighted sum-rate maximization approach.

Index Terms—Two-way relaying, cooperative network, distributed beamforming, power control, SNR balancing.

I. INTRODUCTION

Recently, devising bandwidth-efficient collaborative schemes for two-way relay networks have been the focus of several pioneer studies. Often referred to as bidirectional relaying, these schemes aim to establish a two-way communication between two transceivers with the help of one or several relay nodes. Among these schemes, the so-called multiple access broadcast channel (MABC) approach, is known to offer a high bandwidth efficiency as it requires only two phases (time slots) to accomplish the exchange of two information symbols between the two transceivers [1]. In the first phase of this scheme, the two transceivers simultaneously broadcast their signals over the same wireless medium to the relays by using a multiple access scheme to transmit their information. In the second phase, the relays simultaneously broadcast a processed version of their received signals back to the two transceivers. In this protocol, the interference cancelation may not be needed at the relays as the transceivers can eliminate self-interference [1]–[12]. As such the signal-to-noise ratio (SNR) can be used as a measure of quality of service (QoS).

In this work, we consider a two-way relay network consisting of two transceivers and multiple relays, each with a single antenna. We study the joint optimal distributed beamforming and power allocation design for two-way relaying under the total power budget constraint. We first establish the achievable beamforming rate region for the network considered. This region is obtained by characterizing the achievable beamforming SNR region, whose boundary is then shown to correspond, point-to-point, to that of the achievable rate region. Furthermore, we show that any boundary point of the rate region can be attained by using the same optimal relay beamforming weights, but with different values of transmit powers at the two transceivers.

We next obtain the optimal beamforming weights through sum-rate maximization. Interestingly, we show that this sum-rate maximization approach yields the same solution as the max-min fair design approach does. In other words, the two criteria are equivalent for joint optimal beamforming design and power allocation strategy. This is somewhat surprising as one would expect that maximizing the sum-rate results in a water-filling type of solution, which would provide a performance superior to that obtained by a max-min fair design approach in terms of the sum-rate. Our results prove that for two-way linear relaying under a total transmit power constraint, these two approaches produce the same solution. Using this relationship, we then develop a semi-closed-form solution to joint optimal beamforming and power allocation for sum-rate maximization. The proposed semi-closed-form solution provides the relay beamforming weights in a closed form, provided that the transmit power of either of the transceivers is known. We also show that the transmit power of each transceiver can be obtained by solving a one-dimensional optimization problem using a simple bisection search method. Thus, the semi-closed-form solution not only reveals the optimal beamformer structure, but also enables us to convert the original multi-dimensional optimization problem to a one-dimensional optimization problem, regardless of the number of relays, thereby providing a significant computational advantage. Numerical examples demonstrate that the computational complexity of the proposed technique is significantly lower than the gradient-based technique developed in [13] for the max-min fair design approach. With the obtained optimal beamforming weights, we determine the optimal transmit

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powers to achieve any point on the boundary of the rate region through weighted sum-rate maximization. Lastly, we discuss the beamforming design using other weighted max-min fair criteria. We need to emphasize that our results on the optimal distributed beamforming design are under the assumption that the underlying relaying protocol is MABC under a total power budget and the relays employ AF protocol.

In [2], a two-way beamforming technique has been presented for a three-node relay network, consisting of two transceivers and one relay. It is therein assumed that the relay is equipped with multiple antennas which allow spatial processing capabilities at the relays. In our problem, we consider multiple single-antenna relays, and thus, do not assume any spatial processing capabilities at any relay or among relays. For multiple relays, the authors of [8] study the problem of designing an optimal two-way network beamformer to maximize the sum-rate under the total relay transmit power constraint with fixed transceiver transmit powers. In [14], the achievable beamforming rate region has been characterized for a two-way multi-relay network under both individual and total relay power constraints. Our work differs from [8] and [14] in that we fix the total power consumed in the network, and jointly optimize the beamforming weights at the relays and the power distribution among the relays and two transceivers, while the authors of [8] and [14] assume that the transceiver transmit powers are fixed and they optimize only the beamforming weights at the relays. In [13], a total transmit power minimization technique under quality of service constraints as well as a max-min fair design approach under a total transmit power constraint were studied. It was proven in [13] that using such design approaches to devise jointly optimal distributed beamforming and power allocation yields a unique solution. Gradient based solutions were also proposed. Different from [13], in this work, we focus on 1) characterizing the achievable beamforming rate region, 2) exploring the relationship between sum-rate maximization and max-min fair design techniques, 3) obtaining semi-closed-form solution for the optimal relay beamforming weights, 4) finding the optimum relay beamforming weights and transceiver transmit powers corresponding to any point on the boundary of the rate region, and 5) solving the weighted max-min fair beamforming design problems in terms of rate or SNR.

The study in [15] focuses on comparing various protocols for two-way relaying, including the MABC AF protocol where fixed power allocation is assumed. The authors of [15] derived an upper bound for the rate region of the MABC AF relaying and compared that with that of MABC decode-and-forward relaying scheme. In our work, we focus on the MABC AF protocol, with the same assumptions except we set to find the optimal beamformer under total power constraint, which allows us to maximize the rate region with optimized power allocation under the MABC AF protocol.

Throughout this paper, we use the following notations: $(\cdot)^T$, $(\cdot)^*$, $(\cdot)^H$ represent the transpose, the complex conjugate, and the Hermitian operators, respectively; diag$(a)$ yields a diagonal matrix whose diagonal entries are given by the elements of vector $a$, and diag$(A)$ stands for a vector whose entries are the diagonal elements of the matrix $A$. We use $\angle z$ and $|z|$ to denote the phase and the amplitude of a complex number $z$, respectively. The notation $a \succeq 0$ requires all the entries of the vector $a$ to be non-negative, and $A \succeq (\preceq) 0$ means that the matrix $A$ is non-negative (non-positive) definite. We also use $\lambda_{\text{max}}(\cdot)$ and $\mathcal{P}\{\cdot\}$ to denote the maximum eigenvalue and the principal eigenvector of a matrix, respectively.

### II. Preliminaries

Consider a wireless relay network of two transceivers and $n_r$ relay nodes as shown in Fig. 1. In this network, the two transceivers employ $n_r$ relay nodes to establish a bidirectional communication link. We herein study an amplify-and-forward relay signaling to design a two-hop MABC-based two-way relaying scheme. It is well-known that in the absence of any direct link between the two transceivers, the MABC approach is rate-optimal under a two-hop network setup. If a direct link exists between the two transceivers, the two-hop MABC may not be always optimal. This scheme consists of two steps, where in the first step, the two transceivers simultaneously transmit their data to all relays, and in the second step, the relay re-transmits an amplified and phase-steered version of the received signal towards the two transceivers. In our work, the relays are not required to decode any messages as they relay on amplify-and-forward relaying protocol.

Throughout the paper we use the following assumptions and notations:

**A1.** All nodes in the network are equipped with a single antenna for both transmitting and receiving data. As such, local spatial processing capabilities do not exist at the relays or at the transceivers.

**A2.** A flat fading scenario is assumed. We represent the channel between the $i$th relay and the $k$th transceiver as $f_{ik}$, for $i = 1, 2, \ldots, n_r$ and $k = 1, 2$.

**A3.** Both transceivers have the perfect knowledge of both channel vectors $f_1 = [f_{11} f_{12} \ldots f_{1n_r}]^T$ and $f_2 = [f_{21} f_{22} \ldots f_{2n_r}]^T$.

**A4.** The noise process is assumed to be zero-mean and spatially white with variance of 1.

**A5.** We assume that the whole network is to operate under a total power budget.

From a network design point of view, setting a total power constraint is valuable for network planning as it allows
control and/or optimize the total power consumed in the whole network. In addition, such a total transmit power constraint provides a guideline for how to set individual relay powers. As was shown in [13], for SNR balancing, the relays will collectively consume half of the available total transmit power. It is reasonable to assume that each relay on average consumes $1/n_r$ fraction of the half of the total power budget. This argument is particularly correct for symmetrical network setups or when the relays are moving randomly in the environment. In such a scenario different relay channels appear to be drawn from the same i.i.d. distribution. For all these reasons, this constraint has been adopted in the literature for performance analysis and optimal design [13], [16]–[18].

Denote $x = [x_1, \ldots, x_n]^T$ the $n_r \times 1$ complex vector of the received signals at the relays. It is given by

$$x = \sqrt{p_1}f_1s_1 + \sqrt{p_2}f_2s_2 + \nu$$  \hspace{1cm} (1)

where $p_k$ and $s_k$ are, respectively, the transmit power and the information symbol from Transceiver $k$, for $k = 1, 2$, with $E[|s_1|^2] = E[|s_2|^2] = 1; \nu$ is the $n_r \times 1$ complex noise vector at relay receivers, and $E[\nu
u]^T = I$. In the second step, the $i$th relay uses the complex beamforming weight $w_i$ to adjust the amplitude and steer the phase of the signal received at the relay. As such, the vector of transmitted signals from the relays can be modeled in baseband as $t = w \odot x$, where $t \triangleq [t_1, t_2, \ldots, t_n]^T$ and $w \triangleq [w_1, w_2, \ldots, w_n]^T$. Transceivers 1 and 2 receive, respectively, the signals $y_1$ and $y_2$ which are given by

$$y_1 = f_1^HTt + n_1$$
$$= \sqrt{p_1}w^HF_1s_1 + \sqrt{p_2}w^HF_1f_2s_2 + w^HF_1\nu + n_1$$  \hspace{1cm} (2)

$$y_2 = f_2^HTt + n_2$$
$$= \sqrt{p_1}w^HF_2f_1s_1 + \sqrt{p_2}w^HF_2f_2s_2 + w^HF_2\nu + n_2$$  \hspace{1cm} (3)

where $F_i \triangleq \text{diag}(f_i)$, for $k = 1, 2$, $n_1$ and $n_2$ are the corresponding measurement noises with $E[|n_1|^2] = E[|n_2|^2] = 1$, and we have used the fact that $a^T(b \odot c) = b^T\text{diag}(a)c$, where the vectors $a, b,$ and $c$ are of the same size. According to A3, the two transceivers have the perfect knowledge of all channel coefficients. Thus, they both can obtain the optimal values of $w$, $p_1$, and $p_2$ for any given optimality criterion. It follows from this fact along with A3 that the first term in (2) and the second term in (3), often called self-interference, are known to Transceivers 1 and 2, respectively.

These terms can hence be subtracted from the corresponding signal to produce the residual signals $\hat{y}_1$ and $\hat{y}_2$ given by

$$\hat{y}_1 \triangleq y_1 - \sqrt{p_1}w^HF_1f_1s_1$$
$$= \sqrt{p_2}w^HF_1f_2s_2 + w^HF_1\nu + n_1,$$

$$\hat{y}_2 \triangleq y_2 - \sqrt{p_2}w^HF_2f_2s_2$$
$$= \sqrt{p_1}w^HF_2f_1s_1 + w^HF_2\nu + n_2.$$

These residual signals are then used by their respective transceivers to detect the corresponding desired information symbol.

A max-min fair approach for the optimal network beamformer design is proposed in [13], where the smaller received SNR of the two transceivers is maximized subject to a total transmit power constraint. That is, the following optimization problem is solved:

$$\max_{p \geq 0,w} \min(SNR_1,SNR_2)$$
subject to $P_T \leq P$  \hspace{1cm} (4)

where $p = [p_1, p_2]^T$, $P_T$ is the total power consumed in the whole network (i.e., the summation of the transceivers’ powers and the power consumed by all the relay nodes), $P$ is the maximum total transmit power budget, and $SNR_k$ is the ratio of the desired signal power to the noise power at the $k$th transceiver, for $k = 1, 2$.

Here, we briefly review the results obtained by this approach which will be used in the later sections. It is shown in [13] that at the optimum, $SNR_1 = SNR_2$ holds true, thus leading to an SNR balancing result. Hence, the max-min optimization problem in (4) is equivalent to

$$\max_{p \geq 0,w} SNR_1$$
subject to $SNR_1 = SNR_2$ and $P_T \leq P$.  \hspace{1cm} (5)

It can be readily shown that [13]

$$SNR_1 = \frac{p_2w^HH_1h_1^Hw}{1 + w^HD_1w}, \quad SNR_2 = \frac{p_1w^HH_2h_2^Hw}{1 + w^HD_2w}$$  \hspace{1cm} (6)

where $D_1 \triangleq F_1H_1^H$, $D_2 \triangleq F_2H_2^H$, and $h \triangleq [h_1 h_2 \ldots h_n]^T$. Also, the total transmit power consumed in the whole network is given by [13]

$$P_T = p_1(1 + w^HF_1H_1w) + p_2(1 + w^HF_2H_2w) + w^HW.$$  \hspace{1cm} (7)

The optimization problem in (5) is proven to be equivalent to

$$\max_w \frac{(P - w^HW)w^HH_1h_1^Hw}{2(1 + w^HD_1w)(1 + w^HD_2w)}.$$  \hspace{1cm} (8)

Let $w_i = \alpha_i e^{j\theta_i}$, for $i = 1, \ldots, n_r$. It has been proven in [13] that the phase of the optimal beamforming weight $w_i^*$ has to compensate the phase of the $i$th entry of $h$, which is equal to the aggregated phase of the channel coefficients between the $i$th relay and the two transceivers, i.e., the optimal value of $\theta_i$ is given by $\theta_i^*=\angle f_1 + \angle f_2$. Consequently, the vector of amplitudes of entries of $w$, denoted as $\alpha \triangleq [\alpha_1 \alpha_2 \ldots \alpha_n]^T$, can be obtained by solving the following maximization:

$$\max_{\alpha \geq 0} \frac{(P - \alpha^T\alpha)\alpha^Tbh^T\alpha}{2(1 + \alpha^TD_1\alpha)(1 + \alpha^TD_2\alpha)}.$$  \hspace{1cm} (9)

where $b \triangleq [b_1 b_2 \ldots b_n]^T$ and $b_i$ is the amplitude of the $i$th entry of $h$. The optimization problem in (9) is non-convex, and obtaining a closed-form solution may not be possible. However, it has been shown in [13] that (9) has a unique solution.

Let $\gamma_{\text{max}}(P)$ denote the maximum achievable value of $SNR_1$ (or that of $SNR_2$) under the SNR balancing approach for the total transmit power $P$. That is

$$\gamma_{\text{max}}(P) \triangleq \max_{\alpha} \frac{(P - \alpha^T\alpha)\alpha^Tbh^T\alpha}{2(1 + \alpha^TD_1\alpha)(1 + \alpha^TD_2\alpha)}.$$  \hspace{1cm} (10)
Differentiating the objective function in (10) and equating it to zero yields
\[ Q(\alpha)\alpha = \frac{1}{\zeta_3(\alpha)} b \]  
(11)
where \( Q(\alpha) \triangleq \frac{1}{\zeta_1(\alpha)} C_1 + \frac{1}{\zeta_2(\alpha)} C_2 + \frac{1}{\zeta_4(\alpha)} I \), and \( \zeta_1(\alpha) \triangleq 1 + \alpha^T D_1 \alpha \), \( \zeta_2(\alpha) \triangleq 1 + \alpha^T D_2 \alpha \), \( \zeta_3(\alpha) \triangleq \alpha^T b \), and \( \zeta_4(\alpha) \triangleq \alpha^T \alpha \).

Lemma 1: The nonlinear equation in (11) has only one solution for \( \alpha \geq 0 \), and therefore, the objective function in (9) has one unique maximum for \( \alpha \geq 0 \).

Proof: See [13].

Lemma 1 guarantees that a steepest descent type of algorithm can be used to calculate the global unique solution to (9).

III. ACHIEVABLE RATE REGION

In this section, we first obtain the achievable rate region through finding the achievable SNR region for our two-way relay beamforming problem. In Section IV, we explore the properties of this region to obtain the design parameters such that the sum-rate is maximized. To find the achievable SNR region under a total transmit power budget \( P \), we first find the maximum value of SNR2 for any given value of SNR1. This is given by the following optimization problem:

\[ \max_{p \geq 0, w} \text{SNR}_2 \]
subject to \( P_T \leq P \) and \( \text{SNR}_1 = \gamma \).  
(12)
Using (6) and (7), we can rewrite the maximization in (12) as

\[ \max_{p \geq 0, w} \frac{p_1 w^H h h^H w}{1 + w^H D_2 w} \]
subject to \( \frac{p_2 w^H h h^H w}{1 + w^H D_1 w} = \gamma \)
and \( p_1 (1 + w^H D_1 w) + p_2 (1 + w^H D_2 w) + w^H w \leq P \).  
(13)
From the first constraint in (13), we can obtain \( p_2 \) in terms of \( w \) as \( p_2 = \frac{\gamma (1 + w^H D_1 w)}{w^H h h^H w} \). Therefore, we can eliminate \( p_2 \) from (13) and rewrite the maximization in (13) as

\[ \max_{p \geq 0, w} \frac{p_1 w^H h h^H w}{1 + w^H D_2 w} \]
subject to \( \frac{\gamma (1 + w^H D_1 w)(1 + w^H D_2 w)}{w^H h h^H w} + p_1 (1 + w^H D_1 w) + w^H w \leq P \).  
(14)
Note that at the optimum, \( P_T = P \) holds true meaning that the inequality constraint in (14) will be satisfied with equality. Otherwise, we can increase the optimal \( p_1 \) such that the inequality constraint is satisfied with equality. This will consequently increase the objective function in (14), thus contradicting the optimality claim. Therefore, the inequality constraint has to be satisfied with equality. Using this fact, we write \( p_1 \), in terms of \( w \), as

\[ 0 \leq p_1 = \frac{P - w^H w}{1 + w^H D_1 w} - \frac{\gamma (1 + w^H D_2 w)}{w^H h h^H w}. \]  
(15)
With (15), we rewrite (14) as

\[ \max_w \frac{(P - w^H w) w^H h h^H w}{(1 + w^H D_1 w)(1 + w^H D_2 w)} - \gamma \]
subject to \( \frac{(P - w^H w) w^H h h^H w}{(1 + w^H D_1 w)(1 + w^H D_2 w)} \geq \gamma \).  
(16)
Note that in (16), the objective function is the same as the constraint function. We can either ignore the constraint and solve this optimization problem and then check whether the solution is feasible or not, or choose \( \gamma \) such that the constraint is always feasible. Interestingly, the first term of the objective function in (16) is exactly twice the objective function in (8). Therefore, its maximum is given by \( 2\gamma_{\text{max}}(P) \), where \( \gamma_{\text{max}}(P) \) is given as in (10). Since the second term of the objective function in (16) does not depend on \( w \), for any given \( \text{SNR}_1 = \gamma \leq 2\gamma_{\text{max}}(P) \), the maximum value of \( \text{SNR}_2 \), denoted by \( \text{SNR}_{2,\text{max}} \), is then given by

\[ \text{SNR}_{2,\text{max}} = 2\gamma_{\text{max}}(P) - \gamma. \]  
(17)
Note that \( \text{SNR}_{2,\text{max}} \) is achieved when \( P_T = P \). To achieve any values of \( \text{SNR}_2 < \text{SNR}_{2,\text{max}} \), one can decrease the transmit power \( p_1 \) from its optimal value (see (6)) without violating the total transmit power constraint. Based on this observation, we obtain the achievable SNR region, denoted by \( R_s(P) \), for the two-way relay beamforming problem as

\[ R_s(P) = \{(\text{SNR}_1, \text{SNR}_2) : \text{SNR}_1 \geq 0, \text{SNR}_2 \geq 0, \text{SNR}_1 + \text{SNR}_2 \leq 2\gamma_{\text{max}}\}. \]  
(18)
Fig. 2 shows the achievable SNR region. With \( R_s(P) \), the achievable rate region, denoted by \( R_c(P) \), can now be obtained accordingly. Specifically, any rate pair \((R_1, R_2)\) with the corresponding SNR pair \((\text{SNR}_1, \text{SNR}_2) \in R_s(P)\) is achievable. This is equivalent to \((R_1, R_2)\) satisfying

\[ R_2 = \frac{1}{2} \log_2 (1 + \text{SNR}_2) \]
\[ \leq \frac{1}{2} \log_2 (1 + \text{SNR}_{2,\text{max}}) \]
\[ = \frac{1}{2} \log_2 (1 + 2\gamma_{\text{max}}(P) - \text{SNR}_1) \]
\[ = \frac{1}{2} \log_2 (1 + 2\gamma_{\text{max}}(P) - 2^{2R_1}), \]  
(19)
for \( R_1 \in [0, \frac{1}{2} \log_2 (1 + 2\gamma_{\text{max}}(P))] \). Thus, the achievable rate region \( R_c(P) \) is given by the convex hull of all achievable rate pairs \((R_1, R_2)\):

\[ R_c(P) = \text{conv}\left\{ (R_1, R_2) : R_1 \leq \frac{1}{2} \log_2 (1 + 2\gamma_{\text{max}}(P)), R_2 \leq \frac{1}{2} \log_2 (1 + 2\gamma_{\text{max}}(P) - 2^{2R_1}) \right\}. \]  
(20)
where \( \text{conv}\{\cdot\} \) denotes the convex hull operation. Furthermore, we show that the boundary of the rate region \( R_c(P) \) corresponds exactly to the boundary of the achievable SNR region \( R_s(P) \). To do this, we only need to verify that the rate curve corresponding to the boundary of \( R_s(P) \) is concave. That is, we need to prove that

\[ R_2 = \frac{1}{2} \log_2 (1 + 2\gamma_{\text{max}}(P) - 2^{2R_1}) \]
is a concave function of $R_1$, for $\text{SNR}_1 \leq \gamma_{\text{max}}(P)$. This can be easily verified and we will bring the details in Appendix. Therefore, the achievable rate region is finally obtained as
\[
\mathcal{R}_c(P) = \left\{ (R_1, R_2) : R_1 \leq \frac{1}{2} \log_2(1 + 2\gamma_{\text{max}}(P)), \\
R_2 \leq \frac{1}{2} \log_2(2 + 2\gamma_{\text{max}}(P) - 2^{2R_1}) \right\}. \tag{21}
\]

Fig. 3 also illustrates the achievable rate region $\mathcal{R}_c(P)$. Note that in (16), we see that the maximum of the objective function is achieved when the first term is maximized. Therefore, the optimal beamforming weight vector to achieve any point on the boundary of the achievable SNR region $\mathcal{R}_c(P)$ (or equivalently, on the boundary of the achievable rate region $\mathcal{R}_c(P)$) is given by
\[
\mathbf{w}_{\text{opt}} = \arg \max_{\mathbf{w}} \frac{(P - \mathbf{w}^H \mathbf{D}_1 \mathbf{w}) \mathbf{h} \mathbf{h}^H \mathbf{w}}{(1 + \mathbf{w}^H \mathbf{D}_1 \mathbf{w})(1 + \mathbf{w}^H \mathbf{D}_2 \mathbf{w})} \tag{22}
\]
which is only a function of the channel coefficients and the total power budget. Thus, we conclude that we arrive at the same optimal $\mathbf{w}_{\text{opt}}$, up to a phase rotation, to achieve any point on the boundary of $\mathcal{R}_c(P)$. Therefore, different SNR (or rate) pairs on the boundary are attained by using the same optimal beamforming weight vector $\mathbf{w}_{\text{opt}}$, but different values of transmit powers $p_1$ and $p_2$ at the two transceivers. Once $\mathbf{w}_{\text{opt}}$ is obtained as in (22), it cannot be changed arbitrarily. Note that the optimal $p_1$ and $p_2$ depend on $\mathbf{w}_{\text{opt}}$ and $\gamma$. Given $\mathbf{w}_{\text{opt}}$, to achieve different SNRs on the boundary, $p_2$ has to be a linearly increasing function of $\gamma$ and $p_1$ has to be a linearly decreasing function of $\gamma$.

The intuitive explanation why the weight vector $\mathbf{w}_{\text{opt}}$ is the same for all points on the boundary of the SNR region is that the SNR region is shown to have its boundary described as $\text{SNR}_1 + \text{SNR}_2 = 2\gamma_{\text{max}}(P)$, i.e., all the points on the boundary has the same maximum sum SNR. From the beamforming optimization in (22), we can see that the objective function is essentially the sum SNR under given $\mathbf{w}$. The optimal $\mathbf{w}$ is the one that maximizes the sum SNR, and is only a function of the power budget $P$. Thus, the optimal beamforming vector $\mathbf{w}_{\text{opt}}$ is the same at any point on the boundary.

In the following sections, we will explore the shape of the achievable rate region $\mathcal{R}_c(P)$ to obtain the optimal beamforming weight vector $\mathbf{w}_{\text{opt}}$ and transmit powers $p_1^*$ and $p_2^*$ corresponding to any point on the boundary of $\mathcal{R}_c(P)$.

IV. SUM-RATE MAXIMIZATION

In this section, we first focus on finding the optimal beamforming weight vector $\mathbf{w}_{\text{opt}}$, as well as the optimal transmit powers $p_1^*$ and $p_2^*$ such that the sum-rate is maximized. We will later use this result to derive the optimal transmit powers for all the points on the boundary of $\mathcal{R}_c(P)$.

A. Equivalency to SNR Balancing

Maximizing the sum-rate of the two transceivers under a total power budget $P$ can be written as
\[
\max_{p \geq 0, \mathbf{w}} R_1 + R_2 \\
\text{subject to } P_T \leq P \tag{23}
\]
where $R_1 = \frac{1}{2} \log_2(1 + \text{SNR}_1)$, $R_2 = \frac{1}{2} \log_2(1 + \text{SNR}_2)$. Since $R_1 + R_2 = \frac{1}{2} \log_2(1 + \text{SNR}_1)(1 + \text{SNR}_2)$, sum-rate maximization in (23) is equivalent to the following optimization problem:
\[
\max_{p \geq 0, \mathbf{w}} (1 + \text{SNR}_1)(1 + \text{SNR}_2) \\
\text{subject to } P_T \leq P. \tag{24}
\]

Note that since the SNR pair at the optimum of (24) is bounded to be in the achievable SNR region, it has to be on the boundary of this region. Otherwise, we can find a higher SNR$_1$ with SNR$_2$ fixed (or vice versa), which results in higher value of the objective function in (24). Therefore, denoting the
optimal SNR1 and SNR2 achieved by solving (24) as SNR1^0 and SNR2^0, respectively, they should satisfy
\[ SNR_1^0 + SNR_2^0 = 2\gamma_{\text{max}}(P). \] (25)
Hence, without loss of optimality, we can add the constraint
\[ SNR_1 + SNR_2 = 2\gamma_{\text{max}}(P) \]
subject to \( P_T \leq P \) and \( SNR_1 + SNR_2 = 2\gamma_{\text{max}}(P) \). (26)
Note that
\[ (1 + SNR_1)(1 + SNR_2) \leq \left( 1 + \frac{SNR_1 + SNR_2}{2} \right)^2 \]
where the equality is achieved if and only if \( SNR_1 = SNR_2 \).
Therefore, the objective function in (26) is maximized with \( SNR_1^0 = SNR_2^0 = \gamma_{\text{max}}(P) \). Recall that \( \gamma_{\text{max}}(P) \) is the maximum achievable SNR under the SNR balancing approach in (5). Hence, we conclude that (26), and ultimately the sum-rate maximization problem in (23) is equivalent to the SNR balancing problem:
\[ \max_{p > 0, w} SNR_1 \]
subject to \( P_T \leq P \) and \( SNR_1 = SNR_2 \). (27)
This implies that the sum-rate maximization leads to the SNR balancing solution.

It is well-known that in application to parallel channels, power allocation via sum-rate maximization often leads to a water-filling type of solution where the channel(s) with a better quality receives a larger portion of the total transmit power as compared to a channel with worse quality. These types of solutions are often different from solutions obtained via max-min fair design approach, where user fairness is the primary objective. This is the case for point-to-point communications, as well as one-way relaying. Interestingly, for the two-way relaying beamforming problem, we have shown that, unlike traditional water-filling type of solutions, the optimal power allocation for the sum-rate maximization under total power budget constraint leads to an SNR balancing (or max-min) solution. This means that the transceiver with better channels to the relays will require less transmit power as compared to the transceiver with weaker links to the relays. To the best of our knowledge, so far, such an equivalency between the sum-rate maximization and max-min fair design approaches has not been established for any other communication schemes. In fact, the max-min fair approach is known to often result in a lower sum-rate as compared to sum-rate maximization. Our result shows that this is not the case for two-way relay beamforming with linear processing at the relays.

It is worth noting that the equivalence between the sum-rate maximization and the max-min fair design approaches holds true only under the total transmit power constraint. Such a relationship between these two approaches may not exists under individual power constraint or when the powers of the transceivers are fixed.

B. Semi-Closed-Form Solution

In this subsection, we present a semi-closed-form solution of the optimal beamforming design problem based on the sum-rate maximization introduced in the previous subsection. This is equivalent to deriving a semi-closed-from solution to the SNR balancing problem in (5). With (6) and (7), we rewrite the optimization problem in (5) as
\[ \max_{p > 0, w} \frac{p w^H hh^H w}{1 + w^H D_2 w} \]
subject to \( p_1 (1 + w^H D_1 w) = p_2 (1 + w^H D_2 w) \)
and \( p_1 (1 + w^H D_1 w) + p_2 (1 + w^H D_2 w) + w^H w \leq P \). (28)
Using the first constraint in (28), we eliminate \( p_2 \) in the second constraint and rewrite (28) as
\[ \max_{p_1 \geq 0} \frac{p_1}{p_2} \left( \frac{w^H hh^H w}{1 + w^H D_2 w} \right) \]
subject to \( w^H (2p_1 D_1 + I) w \leq P - 2p_1 \). (29)
Let us introduce the \( n_r \times 1 \) unit-norm complex vector \( \tilde{w} \) and the real scalar \( p \) such that
\[ A^{\frac{1}{2}}(p_1) w = \sqrt{p} \tilde{w} \] (30)
where \( A(p_1) \triangleq 2p_1 D_1 + I \) is an \( n_r \times n_r \) diagonal matrix. We then rewrite the inner maximization in (29) in terms of \( \tilde{w} \) as
\[ \max_{p, \tilde{w}} \frac{p \tilde{w}^H B(p_1) \tilde{w}}{1 + \tilde{w}^H C(p_1) \tilde{w}} \]
subject to \( \| \tilde{w} \|^2 = 1 \), and \( 0 \leq p \leq P - 2p_1 \). (31)
where \( B(p_1) \triangleq A^{-\frac{1}{2}}(p_1) hh^H A^{-\frac{1}{2}}(p_1) \) and the \( n_r \times n_r \) diagonal matrix \( C(p_1) \) is defined as
\[ C(p_1) \triangleq A^{-\frac{1}{2}}(p_1) D_2 A^{-\frac{1}{2}}(p_1) = A^{-1}(p_1) D_2. \] (32)
Note that in order to ensure that the feasible set of the optimization problem in (31) is not empty, we need to have \( p_1 \leq P/2 \). Since, for any value of \( \tilde{w} \), the objective function in (31) is monotonically increasing in \( p \), it is maximized when \( p \) equals to its maximum possible value, i.e., when \( p = \hat{p} \triangleq P - 2p_1 \). As a result, we can simplify the optimization problem in (31) as
\[ \max_{\tilde{w}} \frac{\tilde{p} \tilde{w}^H B(p_1) \tilde{w}}{\tilde{w}^H (I + \hat{p} C(p_1)) \tilde{w}} \]
subject to \( \| \tilde{w} \|^2 = 1 \). (33)
The objective function in (33) is globally maximized if \( \tilde{w} \) is equal to the principal generalized eigenvector of \( (B(p_1), I + \hat{p} C(p_1)) \), or, equivalently, it is equal to the principal eigenvector of the matrix \( (I + \hat{p} C(p_1))^{-1} B(p_1) \). Such a global maximizer of the objective function in (33) can be normalized to satisfy the unit-norm constraint in (33). Let us define \( P(p_1) \) as
\[ P(p_1) \triangleq (I + \hat{p} C(p_1))^{-1} B(p_1) \]
\[ = \left( I + \hat{p} A^{-\frac{1}{2}}(p_1) D_2 A^{-\frac{1}{2}}(p_1) \right)^{-1} \]
\[ A^{-\frac{1}{2}}(p_1) hh^H A^{-\frac{1}{2}}(p_1). \] (34)
Then, the solution to (33) can be expressed as
\[
\tilde{w}_o(p_1) = \kappa(p_1)\mathcal{P}\{\mathbf{P}(p_1)\} \\
= \kappa(p_1)\left(\mathbf{I} + (P - 2p_1)\mathbf{C}(p_1)\right)^{-1}\mathbf{A}^{-\frac{1}{2}}(p_1)\mathbf{h} \\
= \kappa(p_1)\left[\mathbf{A}^{-\frac{1}{2}}(p_1) + (P - 2p_1)\mathbf{A}^{-\frac{1}{2}}(p_1)\mathbf{D}_2\right]^{-1}\mathbf{h}
\]
(35)
where the normalization factor \(\kappa(p_1)\) is to ensure \(||\tilde{w}_o(p_1)|| = 1\) and it is given as
\[
\kappa(p_1) = \sqrt{\mathbf{h}^H\left(\mathbf{A}^{-\frac{1}{2}}(p_1) + (P - 2p_1)\mathbf{A}^{-\frac{1}{2}}(p_1)\mathbf{D}_2\right)^{-2}\mathbf{h}} \\
= \sqrt{\mathbf{h}^H\left(2p_1\mathbf{D}_1 + \mathbf{I}\right)^{-1}\left(\mathbf{I} + (P - 2p_1)\mathbf{D}_2\right)^{-2}\mathbf{h}}.
\]
To obtain the second equation in (35), we have used the fact that \(\mathcal{P}\{\mathbf{a}^H\} = \mathbf{a}\). Consequently, from (30), for any given \(p_1\), the optimal beamforming weight vector is given by
\[
\tilde{w}_o(p_1) = \sqrt{P - 2p_1}\mathbf{A}^{-\frac{1}{2}}(p_1)\tilde{w}_o(p_1) \\
= \kappa(p_1)\sqrt{P - 2p_1}\left[\mathbf{A}(p_1) + (P - 2p_1)\mathbf{D}_2\right]^{-1}\mathbf{h}.
\]
(36)
Combining (29), (33), and (36), the corresponding maximum achievable balanced SNR (i.e., when \(\text{SNR}_1 = \text{SNR}_2\)) is expressed as
\[
\text{SNR}_{\text{max}}(p_1) = p_1(P - 2p_1)\lambda(p_1)
\]
(37)
where \(\lambda(p_1)\) is the largest eigenvalue of \(\mathbf{P}(p_1)\). Since the rank of \(\mathbf{P}(p_1)\) is one, its only non-zero eigenvalue is given by \(\lambda(p_1)\)
\[
= \mathbf{h}^H\mathbf{A}^{-\frac{1}{2}}(p_1)\left[\mathbf{I} + \tilde{\rho}\mathbf{A}^{-\frac{1}{2}}(p_1)\mathbf{D}_2\mathbf{A}^{-\frac{1}{2}}(p_1)\right]^{-1}\mathbf{A}^{-\frac{1}{2}}(p_1)\mathbf{h} \\
= \mathbf{h}^H\left(\mathbf{A}(p_1) + \tilde{\rho}\mathbf{D}_2\right)^{-1}\mathbf{h} \\
= \mathbf{h}^H\left(2p_1\mathbf{D}_1 + \mathbf{I} + (P - 2p_1)\mathbf{D}_2\right)^{-1}\mathbf{h}
\]
(38)
which is non-negative. Substituting (38) into (37), the outer optimization problem in (29) can now be written as
\[
\max_{p_1} \quad p_1(P - 2p_1)\mathbf{h}^H\left(2p_1\mathbf{D}_1 + \mathbf{I} + (P - 2p_1)\mathbf{D}_2\right)^{-1}\mathbf{h} \\
\text{subject to} \quad 0 \leq p_1 \leq P/2.
\]
(39)
We now show that the objective function in (39) is guaranteed to have a unique maximum. Note that the optimization problem in (39) is the same as in (28), and we have shown in Section II that the problem in (28) is the same as that in (8). Through (9) and Lemma 1, it is shown that the objective function in (8) has a unique global maximum and no other local maximum. Thus, we conclude that (39) has a unique maximum. Furthermore, it is straightforward to see that this maximum must be attained at \(p_1 \in (0, P/2)\), as the value of the objective function becomes zero for \(p_1 = 0\), or \(P/2\).

1Note that \((2p_1\mathbf{D}_1 + \mathbf{I})^{-1}\) and \((\mathbf{I} + (P - 2p_1)\mathbf{D}_2)^{-2}\) are diagonal matrices.
2Note that \(\mathbf{A}(p_1)\) and \(\mathbf{D}_2\) are diagonal matrices.
3Here we have used the fact that the only non-zero eigenvalue of \(\mathbf{ab}^H\) is equal to \(\mathbf{b}^H\mathbf{a}\).

Thus, we can find the solution by setting the derivative of the objective function to zero.

Differentiating the objective function in (39) results in a polynomial function of order \(2n_r\). Unfortunately, for \(n_r > 1\), equating this polynomial to zero does not yield a closed-from solution. However, the one-dimensional maximization in (39) can be easily solved using bisection method. Specifically, let us denote the objective function in (39) as \(\phi(p_1)\). The derivative of \(\phi(p_1)\) with respect to \(p_1\) is obtained as
\[
f(p_1) = \frac{\partial \phi(p_1)}{\partial p_1}(\tilde{\rho} - 2p_1)\mathbf{h}^H(2p_1\mathbf{D}_1 + \mathbf{I} + \tilde{\rho}\mathbf{D}_2)^{-1}\mathbf{h} - \\
p_1\tilde{\rho}\mathbf{h}^H(2p_1\mathbf{D}_1 + \mathbf{I} + \tilde{\rho}\mathbf{D}_2)^{-2}(2\mathbf{D}_1 - 2\mathbf{D}_2)\mathbf{h}
\]
where \(\tilde{\rho} = (P - 2p_1)\). Let \(p_1^o\) denote the optimal value of \(p_1\) in (39). From above, we have \(f(p_1^o) = 0\), and \(p_1^o\) can be obtained using a simple bisection technique. With \(p_1^o\), the optimal value of the beamforming weight vector in (36) can be obtain as
\[
\tilde{w}_{\text{opt}} = \tilde{w}_o(p_1^o) \\
= \kappa(p_1^o)\sqrt{P - 2p_1^o}\left[\mathbf{A}(p_1^o) + (P - 2p_1^o)\mathbf{D}_2\right]^{-1}\mathbf{h}.
\]
(40)
Using the second constraint in (28), the optimal value of \(p_2\) is obtained as
\[
p_2^o = \frac{1 + \mathbf{w}_o^H(p_1^o)\mathbf{D}_1\mathbf{w}_o(p_1^o)}{1 + \mathbf{w}_o^H(p_1^o)\mathbf{D}_2\mathbf{w}_o(p_1^o)} p_1^o
\]
(41)
It has been proven in [13] that, at the optimum and for any channel realization, the two transceivers consume half of the total power budget \(P\). This means that
\[
p_2^o = \frac{1}{2}P - p_1^o.
\]
(42)
Using (42), we can rewrite the optimal beamforming weight vector in (40) as
\[
\tilde{w}_o(p_2^o) = \kappa(p_1^o)\sqrt{2p_2^o\left(2p_1^o\mathbf{D}_1 + 2p_2^o\mathbf{D}_2 + \mathbf{I}\right)}^{-1}\mathbf{h}
\]
(43)
where \(\kappa(p_1^o) = \left(\mathbf{h}^H(\mathbf{I} + 2p_1^o\mathbf{D}_1)^{-1}\left(\mathbf{I} + 2p_2^o\mathbf{D}_2\right)^{-2}\mathbf{h}\right)^{1/2}\).
Note that \(\left(2p_1^o\mathbf{D}_1 + 2p_2^o\mathbf{D}_2 + \mathbf{I}\right)^{-1}\) in (43) is a diagonal matrix. Hence, the \(i\)th relay can obtain its own beamforming weight \(w_i(p_1^o)\) using its own local channel information and the knowledge of only two additional parameters, \(\kappa(p_1^o)\) and \(p_2^o\) that is common for all the relays. Therefore, if one of the two transceivers broadcasts \(\kappa(p_1^o)\) and \(p_2^o\) parameters, the \(i\)th relay can use these two parameters along with its own local channel information (i.e., \(f_{i1}\) and \(f_{i2}\)) to obtain its beamforming weight as
\[
w_i(p_1^o) = \kappa(p_1^o)\sqrt{2p_2^o(2p_1^o|f_{i1}|^2 + 2p_2^o|f_{i2}|^2 + 1)^{-1}}f_{i1}f_{i2}
\]
(44)
where we have used the fact that the \(h_i = f_{i1}f_{i2}\). Note that each relay can use (42) to obtain \(p_1^o\) from \(p_2^o\). It is worth mentioning that compared to the scheme proposed in [13], the proposed semi-closed-from solution requires the broadcast of only two parameters to all relays while the method of [13] requires the transceiver to broadcast three parameters to all relays. This is yet another advantage of the proposed solution over that proposed in [13].
Using an approach similar to the above procedure, we can show that if in (28), we eliminate $p_1$, instead of $p_2$, a similar solution as a function of the optimal $p_2^*$ can be obtained as

$$w_o(p_2^*) = \kappa(p_2^*) \sqrt{2p_2^* (2p_1^* D_1 + 2p_2^* D_2 + I)^{-1}} h \tag{45}$$

where $\kappa(p_2^*) \triangleq (h^H (I + 2p_1^* D_1)^{-1}(I + 2p_2^* D_2)^{-1} h)^{-1/2}$. Since the optimal beamforming vector is unique up to a phase rotation (i.e., $w_o(p_2^*) = \omega w_o(p_2^*) e^{j\omega}$ for some $\omega \in \mathbb{R}$), the values in (43) and (45) are equal, leading us to the following equality: $\kappa(p_1^*) \sqrt{2p_1^*} = \kappa(p_2^*) \sqrt{2p_2^*}$.

We would like to point out that the semi-closed-form solution we obtained in (43) (or (45)) expresses the optimal beamforming weight $w_{opt}$ in terms of the optimal transmit power $p_1^*$ (or $p_2^*$) at the maximum sum-rate point. As mentioned in Section III, the optimal beamforming weight $w_{opt}$ is the same (up to a phase rotation) at any boundary point of $R_c(P)$. Therefore, $w_o(p_1^*)$ (or $w_o(p_2^*)$) is also the optimal beamforming weight at any point on the boundary of $R_c(P)$.

The semi-closed-form solution to the sum-rate maximization (or equivalently, SNR balancing) problem is summarized as in Algorithm 1. In this algorithm, $\varepsilon$ determines the maximum iteration tolerance in the achieving the maximum balanced SNR.

As a simple suboptimal solution, we can evenly split the power between the two transceivers. Since at the optimum, $p_1^* + p_2^* = P/2$, we have $p_1^* = p_2^* = P/4$. Although this solution is suboptimal, if different diagonal entries of $D_1$ and $D_2$ are drawn from the same probability density function, the choice of $p_1^* = p_2^* = P/4$ can result in a performance which is on average close to that of the optimal solution, thereby alleviating the need for solving (39). The performance of both (optimal and suboptimal) solutions will be compared via numerical examples. The suboptimal solution to the sum-rate maximization problem is summarized as in Algorithm 2.

Remark: The semi-closed-form solution has a lower computational complexity compared to the gradient based solutions (such as the sequential quadratic programming) as originally presented in [13]. This saving in computational complexity is the result of the fact that the proposed semi-closed-form solution solves a one-dimensional problem as opposed to the method in [13], where a $n_\gamma$-dimensional problem is solved. In Section VII, we demonstrate numerically the computational advantages of the proposed semi-closed-form solution over the gradient based method in [13].

V. GENERALIZATION TO WEIGHTED SUM-RATE MAXIMIZATION

It is worth mentioning that, besides sum-rate maximization, knowing the shape of the achievable rate region allows us to solve the more general problem of weighted sum-rate maximization. Note that using weighted sum-rate as the optimization objective can be viewed as sum-rate maximization under certain fairness constraint set by $\eta$ between the two transceivers. This criterion has been adopted as the utility function for various system/network optimizations [19]–[21].

Algorithm 1 Semi-closed-form solution for sum-rate maximization (or, equivalently, the SNR balancing problem).

Step 1. Set $\varepsilon$ to be any arbitrarily small iteration tolerance, let $p_l = 0$ and $p_u = P/2$, set $k = 1$, $p_1^{(k)} = (p_l + p_u)/2$, and define

$$f(p_1) \triangleq \langle \hat{p} - p_1 \rangle h^H (2p_1^* D_1 + I + \hat{p}D_2)^{-1} h - p_1 \hat{p} h^H (2p_1^* D_1 + I + \hat{p}D_2)^{-2}(2D_1 - 2D_2).$$

Step 2. If $f(p_1^{(k)}) > 0$, then set $p_l = p_1^{(k)}$. If $f(p_1^{(k)}) < 0$, then set $p_u = p_1^{(k)}$.

Step 3. $p_1^{(k+1)} = (p_l + p_u)/2$.

Step 4. If $\left| p_1^{(k+1)} - p_1^{(k)} \right| \lambda(p_1^{(k+1)}) > \varepsilon$, go to Step 2.

Step 5. Set $p_1^* = p_1^{(k)}$ and obtain the optimal value of the weight vector as

$$w_{opt} = w_o(p_1^*) = \kappa(p_1^*) \sqrt{P - 2p_1^* (A(p_1^*) + (P - 2p_1^*)D_2)^{-1}} h.$$  

Step 6. Obtain $p_2^*$ as $p_2^* = p_1^* + p_0^*$.

Algorithm 2 Suboptimal solution to sum-rate maximization (or, equivalently, the SNR balancing problem).


Step 2. Obtain the (suboptimal) value of the weight vector as

$$w_o(p_1^*) = \kappa(p_1^*) \sqrt{P - 2p_1^* (A(p_1^*) + (P - 2p_1^*)D_2)^{-1}} h.$$  

The corresponding optimization problem can be written as

$$\max_{p_0 \geq 0: w} R_1 + \eta R_2 \quad \text{subject to} \quad P_T \geq P. \tag{46}$$

Here, $\eta$ is a constant coefficient denoting the relative weight of the two rates. By adjusting $\eta$, any point on the boundary of the rate region can be achieved. Since the rate pair $\{R_1, R_2\}$ at the optimum should be achievable, we can solve the optimization problem in (46) by finding the maximum $c$, for which the line $R_1 + \eta R_2 = c$ intersects with the boundary of $R_c(P)$. We consider the following three cases:

1) $\eta \in [(1 + 2\gamma_{max}(P))^{-1}, (1 + 2\gamma_{max}(P))]$; This corresponds to the case when the line $R_1 + \eta R_2 = c$ is tangent to the boundary of the achievable rate region, given by $R_2 = \frac{1}{2} \log_2(2 + 2(1 + 2\gamma_{max}(P))^{-1})$ (see Fig. 3). To find this point, we need to solve the following equation

$$\frac{\partial R_2}{\partial R_1} = -\frac{1}{\eta}$$

or, equivalently

$$\frac{2^{R_2}}{2 + 2\gamma_{max}(P) - 2^{R_1}} = \frac{1}{\eta}.$$
Solving for $R_1$ yields
\[ R_1 = \frac{1}{2} \log_2 \left( \frac{2 + 2\gamma_{\max}(P)}{1 + \eta} \right). \]

Note that since $R_1 \in [0, \frac{1}{2} \log_2(1 + 2\gamma_{\max}(P))]$, we verify that the parameter $\eta$ must satisfy
\[(1 + 2\gamma_{\max}(P))^{-1} \leq \eta \leq 1 + 2\gamma_{\max}(P)\]
as indicated by the range of $\eta$ given in this case. The corresponding SNR at Transceiver 1, denoted by $\text{SNR}_1^\eta$, can then be obtained by solving the following equation:
\[ \frac{1}{2} \log_2(1 + \text{SNR}_1^\eta) = \frac{1}{2} \log_2 \left( \frac{2 + 2\gamma_{\max}(P)}{1 + \eta} \right), \]
which yields
\[ \text{SNR}_1^\eta = \frac{1 + 2\gamma_{\max}(P) - \eta}{1 + \eta}. \]

Since $\text{SNR}_1^\eta$ in (47) lies on the boundary of SNR region, the corresponding beamforming weight vector is equal to $\mathbf{w}_o(p_1^\eta)$ given in (43). We can obtain the optimal transmit power $p_2^\eta$ from (6) and (47) as
\[ p_2^\eta = \frac{(1 + \eta)(1 + \mathbf{w}_o^H(p_1^\eta) \mathbf{D}_1 \mathbf{w}_o(p_1^\eta))}{(1 + 2\gamma_{\max}(P)) - \eta} \mathbf{w}_o^H(p_1^\eta) \mathbf{h}^H \mathbf{w}_o(p_1^\eta). \]
The received SNR of Transceiver 2, denoted by $\text{SNR}_2^\eta$, can also be obtained from the fact that on the boundary of the achievable SNR region, we have
\[ \text{SNR}_1^\eta + \text{SNR}_2^\eta = 2\gamma_{\max}(P). \]

Using (47), we obtain $\text{SNR}_2^\eta$ as
\[ \text{SNR}_2^\eta = \frac{2\gamma_{\max}(P) - 1 + \eta}{1 + \eta} \]
and eventually, the corresponding optimal transmit power $p_1^\eta$ of Transceiver 1 can be calculated as
\[ p_1^\eta = \frac{(1 + \eta)(1 + \mathbf{w}_o^H(p_1^\eta) \mathbf{D}_2 \mathbf{w}_o(p_1^\eta))}{(2\gamma_{\max}(P)) - 1 + \eta} \mathbf{w}_o^H(p_1^\eta) \mathbf{h}^H \mathbf{w}_o(p_1^\eta). \]

2) $\eta \in (0, (1 + 2\gamma_{\max}(P))^{-1})$: In this case, it is straightforward to see that the maximum $c$ is obtained when $R_2 = 0$, i.e., $p_2^\eta = 0$, $p_1^\eta = \frac{P}{2}$. In other words, when the relative weight on the rate from Transceiver 1→2 is small enough in the weighted sum-rate maximization, two-way relaying is reduced to one-way relaying with only the direction of Transceiver 2→1 being active.

3) $\eta \in ((1 + 2\gamma_{\max}(P)), \infty)$: In this case, the maximum $c$ is obtained when $R_1 = 0$, i.e., $p_1^\eta = 0$, $p_2^\eta = \frac{P}{2}$. Similar to the previous case, only the direction of Transceiver 1→2 is active.

VI. EXTENSION TO OTHER FAIRNESS-ORIENTED PERFORMANCE METRICS

In the previous section we showed how we can achieve any rate pair on the boundary of the rate region. To identify at which point on the boundary we want to operate, we essentially need to specify a certain user-fairness preference. In some cases, a different performance metric other than weighted sum-rate may be preferred to enforce certain fairness. Here we discuss two different metrics to determine the rate operating point, and thus, to determine the corresponding optimal values of $p_1$ and $p_2$. Instead of the max-min fair design in (4), we can consider the beamforming design under a weighted max-min fair criterion in terms of SNR, expressed as
\[ \max_{p \in \mathbb{R}^+} \min(\text{SNR}_1, \beta \text{SNR}_2) \text{ subject to } P_T \leq P \] (49)
where $\beta$ is a positive constant. It can be readily proven that at the optimum, $\text{SNR}_1 = \beta \text{SNR}_2$ holds true. Hence, the solution to (49) is given by the point where the $\text{SNR}_1 = \beta \text{SNR}_2$ crosses the boundary $\text{SNR}_1 + \text{SNR}_2 = 2\gamma_{\max}(P)$ (see Fig 2). This point is given as
\[ \text{SNR}_1 = \frac{2\beta\gamma_{\max}(P)}{1 + \beta}, \quad \text{SNR}_2 = \frac{2\gamma_{\max}(P)}{1 + \beta}. \]

Since this point is on the boundary of the SNR region, the optimal beamforming weight vector is still given by $\mathbf{w}_o(p_1^\eta)$. Therefore, the corresponding transceiver’s transmit powers are given by
\[ p_1^\beta = \frac{2\gamma_{\max}(P)(1 + \mathbf{w}_o^H(p_1^\eta) \mathbf{D}_2 \mathbf{w}_o(p_1^\eta))}{(1 + \beta)\mathbf{w}_o^H(p_1^\eta) \mathbf{h}^H \mathbf{w}_o(p_1^\eta)}, \quad p_2^\beta = \frac{2\beta\gamma_{\max}(P)(1 + \mathbf{w}_o^H(p_1^\eta) \mathbf{D}_1 \mathbf{w}_o(p_1^\eta))}{(1 + \beta)\mathbf{w}_o^H(p_1^\eta) \mathbf{h}^H \mathbf{w}_o(p_1^\eta)}. \]

Instead of SNR, we can also design beamforming weight using a weighted max-min fair criterion in terms of rate as
\[ \max_{p \in \mathbb{R}^+} \min(R_1, \rho R_2) \text{ subject to } P_T \leq P \] (51)
where $\rho$ is a a positive constant. It can be shown that at the optimum $R_1 = \rho R_2$ holds true. Otherwise, if at the optimum $R_1 < (>) \rho R_2$, then the optimal $p_1$ ($p_2$) can be decreased such that $R_1 = \rho R_2$ and this will not violate the total power constraint. Therefore, the solution can be obtained by finding a point where the line $R_1 = \rho R_2$ crosses the boundary of the achievable rate region, given by $R_2 = \frac{1}{2} \log_2(2 + 2\gamma_{\max}(P) - 2\rho R_1)$, as illustrated in Fig. 3. This leads us to solving the following equation
\[ R_2 = \frac{1}{2} \log_2(2 + 2\gamma_{\max}(P) - 2\rho R_2) \]
or, equivalently
\[ 2^{R_2} + 2\rho R_2 = 2 + 2\gamma_{\max}(P). \]

Unfortunately, the nonlinear equation in (52) does not have a closed-form solution for most value of $\rho$, however it can be solved numerically to find the optimal value of $R_2$, and consequently, that of $R_1$. Then, the corresponding optimal values of $\text{SNR}_1$ and $\text{SNR}_2$ can be obtained. Similar to the previous cases, these optimal values can be used, along with the optimal value for the weight vector, $\mathbf{w}_o(p_1^\eta)$ to determine the optimal values of $p_1$ and $p_2$. The details are omitted for the sake of brevity.
VII. NUMERICAL RESULTS

We consider three networks with 10, 15, and 20 relay nodes. The channel coefficients are modeled as complex zero-mean Gaussian random variables with unit variances. The noise power $\sigma^2$ is assumed to be equal to 0 dBW. For the sake of fairness, the proposed algorithm (presented as Algorithm 1) to compute the semi-closed-form solution, is terminated when $\varepsilon = 10^{-7}$, and the gradient based approach is terminated when the change in the objective function is less than $10^{-7}$. Choosing $\varepsilon = 10^{-7}$ implies that the maximum error in the achieved balanced SNR is $10^{-7}$ which is the same as requiring the change in the objective function for the gradient based method be less than $10^{-7}$.

Note that the computational complexity of our semi-closed-form solution as well as that of the gradient-based solution is $O(n_r)$ per iteration. Therefore, to compare the two algorithms, we use the average number of iterations as the the metric for comparison. Let $N_c$ stand for the average number of iterations for our semi-closed-form solution developed in Section IV-B (Algorithm 1), whereas $N_g$ denotes the average number of iterations for the steepest descent gradient-based counterpart method in [13]. In Fig. 4, we have plotted $N_c$ and $N_g$ versus $P$, in dBW, for different numbers of relays. As shown in this figure, as $P$ is increased, the average number of iterations for our semi-closed-form method changes slightly and this average is almost the same for different numbers of relays. However, the average number of iterations increases significantly for the gradient-based method of [13] with the increase of either the total transmit power $P$ or the number of relays.

In Fig. 5, we have plotted the maximum achievable balanced SNR (i.e., achieved when sum-rate is maximized) versus $P$ for a network of 10 relays where the quality of the channels are the same meaning that all channel coefficients are generated as zero-mean Gaussian random variables with variance $\sigma^2$. In this figure, we have compared the optimal gradient-based method in [13], the optimal semi-closed-form method presented in Algorithm 1, and the suboptimal closed-form technique presented as Algorithm 2. As can be seen from this figure, the two optimal solutions perform identical to each other. Also, the suboptimal Algorithm 2 performs closely to the other two algorithms. Fig. 6 shows the same performance measures for the same network but for a non-symmetric scenario, where the channel coefficients between Transceiver 1 and the relays are generated as zero-mean Gaussian random variables with variance $\sigma^2$, whereas those between Transceiver 2 and the relays are generated as zero-mean Gaussian random variables with variance $\sigma^2$. As shown in Fig. 6, Algorithm 2 suffers around 0.5 dB loss in terms of the maximum achievable balanced SNR as compared to the other two optimal techniques.

VIII. CONCLUSIONS

We have studied the optimal distributed beamforming design for a two-way relay network consisting of two transceivers and multiple relays, all equipped with a single antenna, under a multiple access broadcast relaying scheme. With a total power budget constraint, we have characterized the achievable rate region for such a network, through characterizing the achievable SNR region. Furthermore, we show that all the boundary points of the rate region can be achieved by the same optimal beamforming weight vector, but different values of transmit powers at the two transceivers.

In order to obtain the optimal relay beamforming weights and transmit powers of the two transceivers, we have first considered sum-rate maximization, where we have established the equivalence of the sum-rate maximization approach and the max-min fair design approach under the total power constraint. This conclusion is particularly interesting, as one expects that sum-rate maximization would result in a water-filling type of solution, whose performance is expected to be superior to that obtained by a max-min fair design approach. For the two-way relaying we considered, the two approaches turn out to be equivalent. We further developed a semi-closed-
form solution for the sum-rate-optimal relay beamforming weight vector, which is in a closed form if the transmit power of either of the two transceivers is known. We showed that the transmit powers can be obtained by solving a one-dimensional problem using a simple bisection method. The computational complexity of the proposed technique has been shown, through numerical examples, to be significantly lower than the gradient-based technique developed earlier for the max-min fair design approach.

We have further shown that, for any boundary point on the rate region, the optimal transmit powers of the transceivers can be determined through weighted sum-rate maximization. Lastly, with the obtained achievable rate region, we extended the beamforming design to other weighted max-min fair criteria.

**Appendix A**

**Proof of Concavity of**

\[ R_2 = \frac{1}{2} \log_2 \left( 2 + 2 \gamma_{\max}(P) - 2^{2R_1} \right) \]

It is easy to see that \( \frac{\partial R_2}{\partial R_1} = -\frac{2^{2R_1}}{2 + 2 \gamma_{\max}(P) - 2^{2R_1}} \), and hence

\[
\frac{\partial^2 R_2}{\partial R_1^2} = -\left[ \frac{(4 \ln 2)2^{2R_1}}{2 + 2 \gamma_{\max}(P) - 2^{2R_1}} + \frac{(4 \ln 2)2^{4R_1}}{(2 + 2 \gamma_{\max}(P) - 2^{2R_1})^2} \right] = -\frac{(2 + 2 \gamma_{\max}(P) - 2^{2R_1})^2}{[2 + 2 \gamma_{\max}(P) - 2^{2R_1}]^2} \\
\leq 0.
\]

**References**


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