RECONCEPTUALIZING KNOWLEDGE AT THE MATHEMATICAL HORIZON

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As an activity at the end of a busy school day, Mrs. White asked her Grade 3 students to count how many triangles were in the following figure:

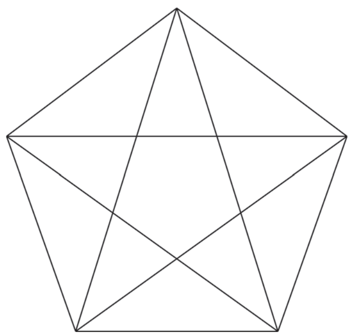

Figure 1. How many triangles?

One group of students appeared to agree that the number of triangles was 32, another group counted 27. The teacher, though she had not yet counted herself, immediately knew that both answers were incorrect. How did she know?

In this article, we analyse the specific mathematical knowledge of teachers such as Mrs. White that emerges in response to students' activity or inquiry. Later in the article, we discuss the particular knowledge acquired by Mrs. White during her undergraduate studies that informed and guided her response in this teaching situation. This example, as well as several others we discuss, sheds new light on the familiar question of which mathematical knowledge is useful and important for teaching. Our focus is on teachers' mathematical knowledge beyond the school curriculum and, in particular, on mathematics learnt during undergraduate studies. We explore what such knowledge may contribute to teaching in terms of relevant subject matter, as well as how such knowledge can shape interaction with students. We introduce a new perspective to the on-going discussions around mathematics for teaching by considering what may lie at the horizon of teachers' mathematical knowledge.

Horizon knowledge – conceived

Mathematical knowledge *of* teachers and mathematical knowledge *in* teaching has attracted wide attention in recent mathematics education research (*e.g.,* Adler & Ball, 2009; Davis & Simmt, 2006; Sowder, 2007). One explicit categorization of teachers' knowledge was introduced by Ball and colleagues (Hill, Ball & Schilling, 2008). It was referred to as "mathematical knowledge for teaching" and presented as an extension of Shulman's (1986) classical categorizations of Pedagogical Content Knowledge (PCK) and Subject Matter Knowledge (SMK). The oval diagram that introduced subcategories of PCK and SMK is often referred to as "the egg".

Hill *et al.*'s (2008) refinement of Pedagogical Content Knowledge included:

- Knowledge of Content and Students,
- Knowledge of Content and Teaching, and
- Knowledge of Curriculum.

Their refinement of Subject Matter Knowledge included:

- Common Content Knowledge,
- Specialized Content Knowledge, and
- Knowledge at the Mathematical Horizon.

This latter category of knowledge at the mathematical horizon has attracted our attention and is of interest in this paper.

To our surprise we found out that while Hill *et al.* explained or defined the majority of what their "egg diagram" categories entailed, there was no explanation of what they meant by mathematical horizon. We started to construct our own meaning of this term. Searching through other related publications, we found the following: "Horizon knowledge is an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (Ball, Thames & Phelps, 2008, p. 403). We wondered how this subcategory of SMK may be different from the subcategory of PCK called "knowledge of curriculum". Further, the notion of horizon knowledge was referenced in Ball (1993), an article that indeed demonstrates the teacher's awareness of what might be future needs of her students, but precedes the work on knowledge categorization.

While working on conceptualizing our view of horizon knowledge we came across more recent work by Ball and Bass – in conference presentations [1] and conference proceedings – that explicitly attends to the notion of horizon knowledge:

> We define horizon knowledge as an awareness – more as an experienced and appreciative tourist than as a tour guide – of the large mathematical landscape in which the present experience and instruction is situated. It engages those aspects of the mathematics that, while perhaps not contained in the curriculum, are nonetheless useful to pupils' present learning, that illuminate and confer a comprehensible sense of the larger significance of what may be only partially revealed in the mathematics of the moment (Ball & Bass, 2009, p. 5).

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Ball and Bass (2009, p. 5) further describe their conception of horizon knowledge as consisting of four elements:

1. A sense of the mathematical environment surrounding the current "location" in instruction
2. Major disciplinary ideas and structures
3. Key mathematical practices
4. Core mathematical values and sensibilities.

However, considering the specific examples provided by Ball and Bass (2009), we see an appropriate fit with the subtitle of their paper: "Knowing mathematics for teaching to learners' mathematical futures" as well as with their claim that "teaching can be more skilful when teachers have mathematical perspective on what lies in all directions, behind as well as ahead, for their pupils, that can serve to orient their navigation of the territory" (p. 12). That is, while attending to *teachers'* mathematical knowledge, they appear to focus on *learners'* horizons. But what about the teachers' horizons?

Teachers' horizon knowledge is, for us, deeply connected to their knowledge of advanced (university or college level) mathematics. In this paper we extend the idea of knowledge at the mathematical horizon by focusing on the teacher and exemplifying what teachers' knowledge "beyond school curriculum" can bring to teaching. Some of these examples are based on our personal experiences, while others are excerpted from conversations with teachers.

Horizon knowledge – extended

Our notion of knowledge at the mathematical horizon (KMH) is related to what Zazkis and Leikin (2010) defined as advanced mathematical knowledge (AMK). Acknowledging that different curricula exist at different times and in different parts of the world, they defined advanced mathematical knowledge as "knowledge of the subject matter acquired during undergraduate studies at colleges or universities" (p. 1). We consider application of advanced mathematical knowledge in a teaching situation as an instantiation of teachers' knowledge at the mathematical horizon. More explicitly, a teacher's use of the mathematical subject matter knowledge acquired in undergraduate studies is recognized as an instantiation of knowledge at the mathematical horizon when such knowledge is applied to a secondary or elementary school teaching situation.

Our view is influenced by the metaphorical definition of horizon as a place "where the land appears to meet the sky". We interpret this as the place where advanced mathematical knowledge of a teacher (the sky) appears to meet mathematical knowledge reflected in school mathematical content (the land). Or, following Felix Klein, it is an advanced perspective on elementary mathematics that is applicable to teaching.

Broader philosophical views of horizon also influence our understanding. In particular, Husserl's notions of inner and outer horizon (Follesdal, 2003) resonate with our description of mathematical horizon as the place where advanced mathematical knowledge meets school curriculum. According to Husserl, when an individual attends to an object, his or her focus centres on the object itself, while in the peripheral of the object lies the rest of the world. As such, the horizon of an object, which includes all the features in the peripheral, may be partitioned into an "inner horizon" and an "outer horizon". Briefly, Husserl's notion of inner horizon corresponds to aspects of an object that are not at the focus of attention but that are also intended. For example, if we were to consider the chair on which one sits, we might attend to its leather back, its swivel motion, the height of its arm rests, *etc*. The inner horizon would thus include all of the other aspects of "chair" that are outside of our focus, either because they are taken for granted, or because they are not yet within our awareness. Such features of "chair" might include the fact that it is for sitting on, or that it is an "office chair" and not a recliner or lounge chair, or that if one were to lean too far back in it one might fall over. In contrast, the outer horizon of an object includes features which are not in themselves aspects of the object, but which are connected to the world in which the object exists. Considering the chair, its outer horizon would include, for instance, the class of objects (furniture) designed for sitting upon.

Further, we can interpret inner and outer horizons of a mathematical object. For example, if we consider the graph of the function $y = 2x^2 + 3$ and attend to its shape (a "stretched" parabola) and its location on the plane (with vertex at (0, 3), contained in quadrants I and IV), then the inner horizon would include all of the aspects of the graph of $y = 2x^2 + 3$ that are outside of our focus. Such features would include the fact that $y = 2x^2 + 3$ has no real roots, that it is symmetric (about $x = 0$), that it is a specific example of a polynomial function with even degree, or that it is differentiable. In contrast, the outer horizon of an object includes features which are not in themselves aspects of the object, but which are connected to the world in which the object exists. With respect to $y = 2x^2 + 3$, its outer horizon includes, for example, the set of conic sections, and the set of functions that are differentiable, even, or concave up. Additionally, the outer horizon of a mathematical object includes much more than generalisations of the specific features exemplified by the object. It also includes the connections between different disciplinary strands and contexts in which the object may exist. In the case of the parabola, there is a connection between its existence in, for example, a calculus context and a geometry context. Differentiability and concavity, symmetry and evenness, are features of the "worlds" of both geometry and calculus and, as such, the connection between the two contexts exists as a feature of the outer horizon of the parabola.

To reiterate, within Husserl's interpretation, an object's inner horizon is composed of specific features of the object itself and includes the attributes of the object that lie in the periphery of our focus. In particular, what exists in the inner horizon of an object is dependent on our choice of focus as we attend to that object. For instance, if we attend to a graph's location and vertex, then other attributes, such as its symmetry, become "out of focus" and thus exist as aspects of the graph's inner horizon. If we were instead to attend to the graph's symmetry, then its specific location in our coordinate system might fade to the periphery of our focus and, as such, would become part of the graph's inner horizon. In contrast, it is not the particular features *of* the object which encompass its outer horizon, but rather features that are *connected* to the object and that embed it in a greater structure. The outer horizon represents the "greater world" in which an object exists. It is independent of focus and consists of the generalities which are exemplified in the particular object. Thus, while symmetry of the graph would lie in its inner horizon, all possible symmetries of all possible graphs would lie in its outer horizon.

Following the metaphorical and philosophical views of horizon, what encompasses the horizon depends on where one stands. That is, the higher one stands, the farther away the horizon is and the more it encompasses. Consequently, that which is "on the horizon" for one person, say a student, may be within reach for another, the teacher.

Subsequently, a teacher's knowledge at the mathematical horizon includes features in both the inner and outer horizons of an object, while only some of those features are accessible to students. The horizon being "farther away" for the teacher enables him or her to see more features and attributes of an object, and to gain a more in-depth appreciation for what exists in the outer world. We see a connection here with Ball and Bass's (2009) description of knowledge at the mathematical horizon as "a kind of peripheral vision" (p. 5), which is in accord with our interpretation of inner horizon and, in particular, with their second element of horizon, major disciplinary ideas and structures, which we interpret as part of the outer horizon. In other words, we view major disciplinary ideas and structures as features of the world in which an object exists, yet which are not in and of themselves features of the object. It is here that we focus much of our attention and analysis, though we acknowledge that Husserl's notions of inner and outer horizons encompass much more than our specific implementation or interpretation does.

In summary, we suggest it is teachers' advanced mathematical knowledge which allow them a "higher" stance and broader view of the horizon with respect to the specific features of the object itself (inner horizon) and with respect to the major disciplinary ideas and structures (Ball and Bass's second element) occupying the world in which the object exists (outer horizon). With this in mind, in what follows we provide several examples of knowledge at the mathematical horizon in specific situations of teaching elementary and secondary school mathematics. These examples focus on the teacher and his or her response to students' work and questions, with the aim of illustrating different, possibly unexpected ways that knowledge beyond the school curriculum can influence a teacher's pedagogical choices in the moment.

Horizon in teaching elementary school mathematics

Example 1

In the introductory example of the counting activity used by Mrs. White in her Grade 3 (ages 8-9) class, the students were counting without any specified system or organized approach and it was not altogether surprising to find inconsistent answers. The teacher, though she had not yet determined the number of triangles herself, immediately knew that both answers were incorrect. She recognized rotational symmetry of order 5 in the figure and, as such, she knew that the number of triangles should be a multiple of 5. With this understanding in mind, she helped students identify different kinds of triangles and where, with each triangle-shape found, there were 5 of the same kind. She led students to catalogue different shapes and account for them systematically. What was intended initially as an activity for the last few minutes of a school day turned to purposeful work of identifying congruent shapes.

The ideas of symmetry and divisibility appear briefly in elementary school mathematics. However, it was during her university course "Mathematics for Elementary Teachers", taken as part of her teacher education program, that Mrs. White acquired an understanding of rotational symmetry and its order. As a result, she was able to make a connection to divisibility, which we consider an example of her knowledge at the mathematical horizon. The rotational symmetry of the figure, though not the focus of the activity, was recognized by Mrs. White and so was an accessible aspect of her inner horizon. The connection between the symmetry of the figure and divisibility are features of the outer horizon. This horizon knowledge was implemented in her teaching by directing students toward systematic counting and cataloguing of different shapes, which can be viewed as instantiations of the third element of Ball and Bass's (2009) interpretation of horizon – key mathematical practices.

Example 2

Mr. Green's Grade 5 (ages 10-11) students were learning about factors and multiples. As one of the tasks, he asked the students to identify all the factors of the number 180. Observing the method by which the students were listing factors – sporadically, in order, by pairs – provided him with insight into his students' understanding. He planned to have a discussion about efficient ways of finding all the factors.

Before completing the list, Mr. Green made a note for himself that the number of factors should be 18. Without identifying what was missing on his students' lists, he suggested to several of them to keep looking for more, while asking others to explain why they believed their list was complete. The fact that 180 has 18 factors was knowledge acquired by Mr. Green while taking a discrete mathematics course at college. He recognized the prime factorization of 180 as $2^2 \times 3^2 \times 5$ and, based on the fundamental principle of counting, concluded that the number of factors was $3 \times 3 \times 2 = 18$. (If p appears with exponent k in the prime factorization of n, then there are $k+1$ possible exponents of p in a factor of n, which are $0, 1, \ldots, k$.) Mr. Green did not intend to teach his students the fundamental principle of counting at this time. However, this knowledge at the mathematical horizon related to a major disciplinary idea and structure of numbers, which we interpret as a feature of the outer horizon, helped him in guiding instruction on identifying factors.

Horizon in teaching secondary school mathematics

Example 3

Miss Mauve's Grade 12 (ages 17-18) students had just finished a unit on inverse functions. In her students' work, Miss Mauve observed several instances of confusion in notation which led, among other errors, to miscalculations. Some of her students were writing $1/f(x)$ where they meant $f^{-1}(x)$ and she suspected that students were unclear as to when the reciprocal of a function was, or was not, also its inverse. She concluded that students' inappropriate use of notation was a misgeneralization of previous work with negative exponents, where, for example, 3^{-1} was defined as ⅓.

Miss Mauve decided to spend time clarifying this confusion. She referred to students' experiences with the reciprocal and inverse of numbers, noting that the reciprocal of a number depends on the operation of multiplication, but that the inverse of a number can refer to its additive inverse or its multiplicative inverse (the latter is referred to as the reciprocal). It was during a university course in group theory that Miss Mauve acquired an understanding of the inverse of a group element with respect to the particular operation of that group, a major disciplinary idea of mathematics, which is an aspect of the outer horizon when considering functions. Miss Mauve used this knowledge at the mathematical horizon to help her address her students' confusion and was able to make the idea of inverse with respect to an operation accessible to students without using the terminology of group theory. Her appreciation of group structure and her understanding of its relevance to the specific case of inverse functions are examples of her knowledge at the mathematical horizon, while her explanation in terms accessible to students speaks to her pedagogical content knowledge. A similar instance of confusion and resolution was reported in Zazkis and Zazkis (2011), where a teacher used her understanding of group theory to help her student interpret the meaning of an exponent of negative one in different contexts.

Example 4

During a lesson on applications of derivatives, Mrs. Violet's pre-calculus students were given a set of problems in which they were to calculate derivatives of various functions. The lesson was designed to reinforce calculation techniques through application to standard word problems. The students were unfamiliar with limits, as it was not part of the course curriculum.

As the class worked on their exercises, one student noticed when working with the sphere and circle, that the derivative of the volume formula yielded the formula for surface area and the derivative of the area formula yielded the formula for circumference. That is, $\frac{dV}{dr} = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$ and $\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = 2\pi r$. After class, the student stayed behind to ask why this relationship held for the sphere and the circle and not in other cases such as with the cube and square.

The connection between surface area and volume is one that Mrs. Violet made during a university calculus course. She recalled a geometric representation for the derivative of the area of a circle and was aware of an analogous argument for the derivative of a sphere's volume. Mrs. Violet understood the significance of the diagram shown in Figure 2 and knew that the derivative of the area is defined as:

$$\lim_{h\to 0}\frac{\pi(r+h)^2 - \pi r^2}{h} = \lim_{h\to 0}(2\pi r + \pi h) = 2\pi r.$$

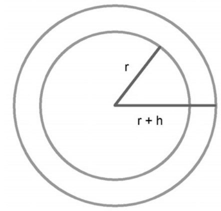

Figure 2. A geometric representation for the derivative of the area of a circle.

Here, $\pi(r+h)^2 - \pi r$ is the difference in area between the circle with radius $r + h$ and the circle with radius r, that is, the area of the ring of width h around the circle with radius r. The change in this difference approaches the circumference of the inner circle as h approaches zero. Similarly, the derivative for the volume is defined as:

$$\lim_{h\to 0}\frac{\frac{4}{3}\pi(r+h)^3 - \frac{4}{3}\pi r^3}{h} = \lim_{h\to 0}\left(4\pi r^2 + 4\pi rh + \frac{4}{3}\pi h^2\right) = 4\pi r^2.$$

Though constructing a frame of width h around a square does not work out as nicely as a ring of width h around a circle, Mrs. Violet was able to refine the diagram in order to get the desired outcome. From her diagram, shown in Figure 2, the similarity with the cases of a circle and a sphere is clear:

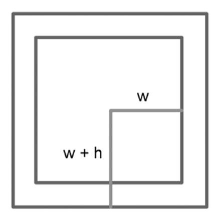

Figure 3. A geometric representation for the derivative of the area of a square.

In this refinement of Mrs. Violet's, w is equal to half the length of one side, and as such, the perimeter of the square is $4(2w) = 8w$. The derivative of the area of the square can thus be written as:

$$\lim_{h\to 0}\frac{4(w+h)^2 - 4w^2}{h} = \lim_{h\to 0}(8w + 4h) = 8w.$$

Similarly, the derivative of the volume of the cube can be expressed as:

$$\lim_{h\to 0}\frac{8(w+h)^3 - 8w^3}{h} = \lim_{h\to 0}(24w^2 + 24wh + 8h^2) = 24w^2,$$

where $24w^2 = 6(2w)^2$ gives the surface area of a cube with side length $2w$.

While it was beyond the scope of the lesson to introduce the definition and calculation of limits to the student, Mrs. Violet used her knowledge of limits to give an intuitive and geometric explanation for why this relationship holds. Her explanation used the above diagrams to illustrate derivative as a rate of change that relates area to the shape's boundary and focused on an analogy between a circle's radius and the length denoted as *w*. Particular attention was paid to *what* changes in the case of the square or cube, noting that in the familiar case it is the length of one side which is taken as the variable, rather than half the length as is the case with Mrs. Violet's analogy. It was knowledge of mathematics acquired in her university studies that heightened Mrs. Violet's awareness of the important observations her student had made and of the potential connections that might result. In particular, the connection between calculus and geometry – an aspect of the outer horizon of derivative and an element of Mrs. Violet's knowledge at the mathematical horizon – gave the teacher insight regarding her student's curiosity. This connection was paramount for illustrating for her student the relationship between area and perimeter (as well as volume and surface area), a major disciplinary idea and hence also a feature of the outer horizon.

Interestingly, access to features of the outer horizon was made available despite side-stepping an explanation of limits. The definition of limit, which Mrs. Violet drew on for her own understanding, is a specific feature of derivatives that exists outside the focus of the particular question and so is an aspect of its inner horizon. While there is an important connection between the inner and outer horizon of derivatives as they relate to the interpretation of limits, it was features of the outer horizon which were deemed to be more accessible to the student. Setting derivatives in the broader world of geometric interpretations of calculus gave access to new information regarding the relationships between volume and surface area, area and perimeter, and the corresponding computations.

How can knowledge at the mathematical horizon be acquired?

In providing their perspective on horizon knowledge, Ball and Bass (2009) commented that "we do not know how horizon knowledge can be helpfully acquired and developed" (p. 11). What we have exemplified above suggests one necessary (though not sufficient) requirement for the development of knowledge at the mathematical horizon: engaging in learning mathematics. This requirement is consistent with Watson's claims that teachers' own mathematical studies may impact their teaching practice. Watson (2008) suggests that experience in mathematics at an advanced level, both in terms of concepts and combining concepts, in addition to analyzing complex mathematical statements to uncover familiar structures, are useful and important aspects of teacher education that lend themselves to effective pedagogical decisions. Similarly, Potari *et al.* (2007) found that rich personal subject knowledge corresponded to greater ease and effectiveness in interpreting and developing students' ideas. In the examples presented above, teachers' knowledge at the mathematical horizon included both an advanced understanding of specific concepts learned at university (inner horizon), but also a broader understanding of the connections between concepts (outer horizon).

While teaching is unimaginable without subject matter knowledge, there is no agreement on what depth or breadth of knowledge is essential. Begle's (1979) classic research that showed that learning more mathematics, as identified by the number of courses taken, is not sufficient for successful teaching. However, "not sufficient" is often misinterpreted as "not essential". The infamous example of Ms. Daniels (Borko *et al.,* 1992) – a teacher who was exempt from a course on teaching mathematics because she had completed a calculus course but who could not explain correctly division by a fraction – provided a rationale for more extended training in mathematical instruction. Of course, we have nothing against the necessity of extended training in mathematical instruction. However, had Ms. Daniels (or her clone) taken a course in abstract algebra, she would have likely enhanced her understanding in the following manner: that division is not a separate operation but a reference to multiplication by an inverse, and that "flipping" the numerator and denominator of a fraction creates an inverse element to an original fraction. She would also then be able to connect the idea of inverse to division by zero: division by zero is undefined because zero has no multiplicative inverse. Such a group-theoretic perspective enables a teacher to put two problematic issues in elementary mathematics – division by zero and division by a fraction – under the same umbrella and so enhance personal understanding. This perspective may not be conveyed to elementary school students but, in our view, it is an example of horizon knowledge, as organized in a disciplinary structure.

Indeed, the relevance of a group-theoretic perspective can be seen in the above examples, one of which explicitly refers to group theory (example 3), while another refers to group theory implicitly (cyclical group of order 5 in example 1). We see this as a very profound instantiation of the second component of knowledge at the mathematical horizon identified by Ball and Bass (2009) – major disciplinary ideas and structures. It is not surprising that group theory appears in our examples, as it provides structure for the main concepts of school mathematics: numbers and functions. Such a structural understanding also fosters an understanding of connections *across* disciplinary ideas, a subtle issue that is brought to light through our examples.

Conclusion

We agree with Watson (2008) that extended experiences in learning mathematics "is a good way to deepen and develop mathematical knowledge in and for teaching" (p. 7). While many avenues are possible for such "extended experiences" we suggest that undergraduate courses in mathematics provide a natural one. The four examples presented in this article illustrate how teachers' knowledge beyond school curriculum, specifically the subject matter knowledge acquired in colleges and universities, can contribute to teachers' instructional choices and be potentially beneficial for students' learning. We considered the application of such knowledge (previously referred to as AMK) as an instantiation of teachers' knowledge at the mathematical horizon (KMH). As such, we have extended the scope of "horizon knowledge", a term previously coined by Ball and Bass (2009).

Although we have no intention to measure knowledge at the mathematical horizon, we find it – from the refined perspective provided here – useful to exemplify particular applications of extended knowledge of the subject, or advanced mathematical knowledge, in teaching situations. Although related to the work of Ball and Bass (2009), our notion of knowledge at the mathematical horizon differs from what they describe as "a kind of elementary perspective on advanced knowledge" (p. 10). Rather, we see it as an advanced perspective on elementary knowledge, that is, as advanced mathematical knowledge in terms of concepts (inner horizon), connections between concepts (outer horizon), and major disciplinary ideas and structures (outer horizon) applied to ideas in the elementary school or secondary school curriculum.

Notes

[1] For example, slides of a presentation by Ball and Bass presented at the National Council of Teachers of Mathematics Annual Meeting, Washington D.C., April 23, 2009, available from: www-personal.umich.edu/~dball/presentations/index.html

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Can we use symmetry as a vehicle to teach certain aspects of problem solving, and if so, should we? In what ways can an understanding of symmetry in general (not merely in geometry) assist in the learning of mathematics?
(David Robitaille, FLM 4(3), p. 26)

Hypothesis: The breadth of extent […] of a person's mathematical knowledge is directly related to experience with mathematical variations. A person using a variation-rich environment will have a more extensive mathematical knowledge (within a given general area) than one who has not had such experiences.
(Thomas Kieren FLM 4(1), p. 43)

How can I discern the extent of generality perceived by someone else when looking at a particular case of what I see as a generic example?
(John Mason, FLM 4(3), pp. 24-25)

Is it possible to offer a complete mathematics curriculum in terms of awareness?
(Dick Tahta, FLM 4(1), p. 47)
