CHAPTER 1

PERIODIC MOTION VS. TURBULENT MOTION: SCALING LAWS, BURSTING AND LYAPUNOV SPECTRA

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Developed turbulence is traditionally defined in terms of, and described by, mean quantities, extracted from statistical analysis of measurements and data from simulations. The merit of the statistical approach can hardly be overestimated as it unveils universal laws such as the scaling of the energy spectrum in the universal range and the velocity profile in near-wall flows. Averaging over an ensemble of flow fields or a long time series does, however, obscure the instantaneous properties of the flow. Therefore a study of the dynamical processes that collectively produce the universal laws requires a different approach. We propose to study these processes by means of time-periodic solutions of the Navier–Stokes equation. From the point of view of chaos theory such periodic solutions are expected to be ubiquitous in turbulence. In order to find and analyse them, however, we need to overcome some fundamental difficulties related to the complexity of their bifurcation diagrams and the large number of degrees of freedom. As an example we present several periodic solutions of the Navier–Stokes equations on a triply periodic domain at moderate Reynolds number.
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"The ideas of low-dimensional dynamics and, in particular, chaos has not shed much light on what has often been called the last unresolved problem of classical physics: turbulence. (...) This is more a measure of the depth of the problem which will probably require a combination of innovative ideas for a significant breakthrough to be made. I remain convinced that an essential ingredient of this will be based in nonlinear dynamical systems."

Tom Mullin

1. Introduction

Although a straightforward, generally accepted definition of developed turbulence does not seem to exist, two key elements are certainly part of it: motion on a wide range of spatial scales and coherent structures. The existence of coherent structures, confirmed both by experiment and by simulations, gives us some hope that we can understand turbulence in terms of relatively simple, computationally tractable objects. Rather than to process information about the entire flow field we can concentrate on a number of coherent structures and see if and how they interact and break down into smaller scale structures, as the conventional picture of the energy cascade suggests. For this purpose we need an objective procedure to identify coherent structures and track their behaviour in time. This proves to be a hard problem indeed, and a substantial amount of work in turbulence research is going into this issue.

The common approach to this problem is based on analysis of data in physical space, such as isosurfaces of entropy or velocity correlations. A different approach, more common in mathematical literature, is to think of a space and time dependent flow field as an orbit in a high-dimensional phase space. The study of such orbits lies in the domain of dynamical systems theory. In particular, dynamical systems theory gives us tools to study special solutions, such as equilibrium points, corresponding to stationary flows, and periodic orbits, corresponding to time-periodic flows. In a dissipative system, such as viscous fluid flow, the orbits in phase space settle on an attractor, and for high enough Reynolds number this attractor will be chaotic. Chaos theory tells us that this "turbulent attractor" contains infinitely many unstable periodic orbits. If we study a certain orbit segment, i.e. a finite time series, which corresponds to the formation and breakdown of a coherent structure in physical space, we will find that it is not periodic. However, there is always a periodic orbit close to it. In the words of Henri Poincaré:

"Given the equations (...) and a particular solution one can always find a periodic solution (of which the period may indeed be very long) such that
the difference between the two remains as small as one likes for as long as one likes.”

Thus, if we can find one or more periodic orbits which lie in the turbulent attractor we can use them to study turbulence and its fundamental dynamical processes.

In this chapter we show how this idea can be applied to the study of isotropic turbulence.\(^1\) This work was preceded by a study of Kawahara and Kida into plane Couette flow.\(^2\) In a separate development, Hof et al. found evidence for the relevance of periodic solutions to pipe flow.\(^3\) Although this is all work of the last few years, the application of ideas from dynamical systems theory to fluid dynamics is certainly no novelty. In the study of laminar or weakly turbulent flows this approach has proven rather successful. Helped by a highly symmetric geometry, flows such as Rayleigh-Bénard and Taylor-Couette have been analysed in terms of equilibria and periodic orbits and their bifurcations. Examples can be found, e.g., in Ref. 4 and in Tom Mullin’s chapter in this volume. Typically, the flow is first analysed at low Reynolds number, where it is stationary and laminar. At higher Reynolds number the flow bifurcates as it turns unstable to, e.g., traveling waves, represented by periodic orbits. If we further increase the Reynolds number solutions of increasing complexity are created, representing more complex physics. We might find invariant tori representing modulated waves or homoclinic orbits representing solitary waves. At some point chaos sets in, and by studying the critical bifurcation we can determine some properties of the resulting, intrinsically unpredictable, fluid motion.

When we try to extend this work to high Reynolds number flows encounter several theoretical and practical problems. In the first place, bifurcation analysis does not always turn out as nicely as it does in the celebrated examples mentioned above. A notoriously hard problem occurs in certain shear flows, such as pipe flow and plane Couette. In these cases, the laminar equilibrium solution is stable for all Reynolds number. Thus there is no straightforward way to find any of the more complex, space and time dependent solutions that play a role in the onset of turbulence. Such solutions may be unstable and bifurcate from infinite Reynolds number, which makes them hard to detect. We face another problem if many sub critical bifurcations occur, in which unstable solutions are created. One soon ends up with large numbers of coexisting, unstable equilibria and periodic solutions. There is no easy way to decide which are the most important to track and the system is prone to sudden transitions caused by collisions of global manifolds that cannot be predicted on the basis of local bifurcation analysis.
Thirdly, even if we can track all solutions and study their bifurcations it is not guaranteed that this gives us the necessary information to understand the physical transition to developed turbulence. In isotropic turbulence, for instance, the transition from weak to developed turbulence appears to be gradual rather than related to a particular bifurcation. The derivative of the energy dissipation rate with respect to the viscosity changes when developed turbulence sets in. From the point of view of dynamical systems theory the behaviour is chaotic all along and it seems unlikely that a particular bifurcation scenario could explain the transition.

Bearing in mind these difficulties, we do not attempt to compute the bifurcation diagram at low Reynolds numbers. This diagram is known to involve bifurcating tori\textsuperscript{5,6} which spawn infinitely many periodic orbits in a myriad of scenarios. Instead, we filter periodic orbits directly from a time series obtained in the weakly turbulent regime. Subsequently we continue the periodic orbits to higher Reynolds number. Along the continuation curve we compute the time-mean energy dissipation rate along the periodic orbits and compare it to the values found in turbulence. Thus, we can see if the periodic orbits reproduce the onset of developed turbulence. One of the orbits we study here reproduces the turbulent time-mean values very well and is analysed in some detail. In particular, we compute the local Lyapunov exponents along this orbit and show them to be correlated to the time dependent energy spectrum.

A practical problem with this work is the large number of degrees of freedom we need to take into account. Generally speaking, we need to study systems of millions of differential equations in order to apply tools of dynamical systems theory to turbulence. We used symmetry considerations to bring this number down to manageable proportions. All computations were done on parallel processors in a time frame of months. As faster and bigger computers are developed and smarter algorithms form numerical linear algebra are becoming available, we expect this to be a transient problem. Within years, we will be able to study periodic solutions in flows with fewer (symmetry) constraints and at higher Reynolds number. This might be the key to significant progress in turbulence research.

2. Turbulence, chaos and the cycle expansion

Before we go into the details of isotropic turbulence and our simulations we will explain in some detail the mathematical motivation of this work. It is based on the special role of periodic orbits in chaos theory. At the time of
writing it is impossible to apply this theory to developed turbulence with mathematical rigour. There is no proof of the existence of periodic solutions to the three-dimensional Navier–Stokes equations, nor of the existence of a finite-dimensional attractor for high-Reynolds number flows. Nevertheless, the concept of the cycle expansion is intuitively clear and inspiring.

In the lingo of chaos theory the observation by Henri Poincaré, cited in the introduction, might be rephrased as follows: 

"Unstable periodic orbits lie dense in a chaotic attractor."

Thus, a chaotic orbit can be thought of as composed of infinitely many segments, each of which closely follows a periodic orbit. This observation led Cvitanović and coworkers to the idea that we might replace the time mean of any quantity along a chaotic orbit by a weighed average over the time-mean values of this quantity along periodic orbits. This idea is not unlike the Bolzmann Anzats in statistical physics, in which the time mean is replaced by an ensemble average. The resulting theory is known as cycle expansion theory because it provides us with an approximation of the time-mean value by a sum of terms generated by periodic orbits (cycles). A full explanation of this approach can be found in Ref 7. Here, we will only provide a sketch to illuminate the similarity to our work.

Consider a chaotic dynamical system with an attractor of dimension $D < 3$. In order to write down a cycle expansion for a time-mean quantity $\bar{a}$ we proceed in four steps, illustrated in Fig. 1:

A In a Poincaré intersection plane we see thin, folded filaments, reminiscent of the Hénon attractor. Periodic orbits correspond to fixed points of the iterated Poincaré map, i.e. $P^n(x) = x$, with discreet period $n$. By a smart parameterisation of the filaments we can reduce the problem to the study of a one-dimensional map.

B Now we define a partition on the one-dimensional domain, and translate the dynamics of the continuous-time system into symbolic dynamics.

C If we know the symbolic dynamics we can list all periodic points up to a given symbolic length, i.e. period. Denote the set of $k(n)$ points of period $n$ by $\Gamma^n = \{\gamma^n_i\}_{i=1}^{k(n)}$.

D For each periodic point we compute a weight, $\mu$, that depends on its Lyapunov spectrum. The cycle expansion is now given by

$$\bar{a} = \lim_{n \to \infty} \sum_{i} a_{\gamma^n_i} \mu(\gamma^n_i)$$

where $a_{\gamma^n_i}$ is the time-mean value of $a$ along the periodic orbit $\gamma^n_i \in \Gamma^n$ and $\mu(\gamma^n_i)$ is the weight of that orbit.
Ordering of all periodic points
\( x_P^n(x) = x \) of period \( n \) and the corresponding orbits \( n_i \in \mathbb{N} \).

Time averaged quantities computed as
\[
\bar{a} = \lim_{n \to \infty} \sum_{i} a_i \mu(\gamma_i^n)
\]

Symbolic dynamics

Fig. 1. A schematic representation of the cycle expansion.

These steps are worked out in detail in Ref. 8, in which the cycle expansion is applied to the Kuramoto-Shivashinski equation in a mildly chaotic regime.

There are some technical conditions that the chaotic attractor has to satisfy in order for the prove of validity of this expansion to hold. For our present purposes, however, the most restrictive condition is that on the attractor dimension. In turbulent flows we cannot expect to find such low-dimensional dynamics. The dimension of the turbulent attractor has been estimated to be of order 100 for shear flow\(^9\) and this estimate is probably on the low side. Consequently, an effort has been made to formulate a cycle expansion for high-dimensional chaos. Basically, we can use the expansion as given above, except that there is no way to ensure we know all orbits up to a certain period and the weight of each orbit does not follow from a mathematical theory. An \textit{ad hoc} expansion was applied successfully to the Kuramoto-Shivashinsky equations in a regime of full-fledged spatio-temporal chaos.\(^{10}\) The method by which periodic orbits were localised is explained in Sec. 3.3. It is based on near recurrences, filtered from a long time series, and has an element of chance to it that might seem unsettling to the exact scientist. Statistically speaking, however, following this approach we find
those orbits that are likely to have a large impact on the chaotic dynamics. This idea was formulated in the context of a coupled map lattice in Ref. 11. An interesting prediction in that work is that 

the larger the number of degrees of freedom, the fewer periodic orbits we need to obtain a good estimate of any time-mean quantities.

The cycle expansion for high-dimensional chaos and the statistical approach to this problem are by no means a complete mathematical theory. There are far too many open technical questions and ambiguities. However, the ideas are consistent with all results on periodic orbits in turbulence that we know of, and it would not be the first time that physicists apply successfully a series expansion without a proof of convergence . . .

3. Isotropic turbulence

Consider an incompressible, viscous fluid on a triply periodic domain. Its motion is governed by the Navier–Stokes equation and the divergence-free condition, which are conveniently solved in terms of the Fourier components of velocity, \( \tilde{v}(k, t) \), and vorticity, \( \tilde{\omega}(k, t) \). Energy is dissipated at the rate \( \epsilon = 2\nu Q \), where \( \nu \) is the kinematic viscosity and \( Q = \frac{1}{2} \sum_k |\tilde{\omega}(k, t)|^2 \) is the enstrophy. In order to input energy we fix the Fourier components of vorticity at the smallest wave number \( k_f \) in time. Under the symmetry constraints explained in Sec. 3.2 we have \( k_f = \sqrt{11} \). Like the energy dissipation rate \( \epsilon \), the energy input rate \( \epsilon \) is a function of time. The complexity of the flow is measured by Taylor’s microscale Reynolds number, defined as

\[
R_{\lambda}(t) = \sqrt{\frac{10}{3}} \frac{\mathcal{E}(t)}{\sqrt{Q(t)}}
\]  

(1)

where \( \mathcal{E}(t) = \frac{1}{2} \sum_k |\tilde{u}(k, t)|^2 \) is the energy. The microscale Reynolds number scales as the square root of the geometric Reynolds number commonly employed in shear flows.

We integrate the Navier–Stokes equation numerically, truncating the Fourier series at \(-N/2 \leq k_1, k_2, k_3 \leq N/2\). The nonlinear terms are computed by the spectral method in which the aliasing interaction is suppressed by eliminating all the Fourier components beyond the cut-off wavenumber \( k_{\text{max}} = [N/3] \). The fourth-order Runge-Kutta-Gill scheme with step size \( \Delta t = 0.005 \) is employed for time stepping.
3.1. The number of degrees of freedom

The maximal microscale Reynolds number that can be attained in simulations at a given truncation level is determined by the ratio of the typical size of the largest eddies, $L$, to that of the smallest eddies, $l$. The former is associated with the length scale of the forcing, i.e. $L = 2\pi/k_f$, and the latter with the Kolmogorov length scale, i.e. $l = \eta_k = (\nu^3/\bar{\epsilon})^{1/4}$. In developed turbulence $\bar{\mathcal{E}}$ and $\bar{\epsilon}$ are independent of $\nu$ so that

$$\frac{L}{l} \approx \gamma R_\lambda^{1/2}$$

(2)

where $\gamma \approx 0.13$ does not depend on $\nu$.\textsuperscript{12} As a rule of thumb, we resolve small enough scales if $k_{\text{max}} \approx 1/\eta_k$, i.e. $L/l \approx 2\pi k_{\text{max}}/k_f$.

The lowest resolution that allows us to simulate turbulence is $N = 128$, corresponding to a maximal microscale Reynolds number of $R_\lambda = 67$. At this $R_\lambda$, $\bar{\mathcal{E}}$ and $\bar{\epsilon}$ are approximately constant as a function of $\nu$. However, we capture the inertial range only marginally. At $N = 128$ we have about $3 \times (2\left\lfloor \frac{128}{3} \right\rfloor + 1)^3 \approx 2 \cdot 10^5$ independent Fourier modes. A simulation of this size can be run comfortably on a present day personal computer. As will be explained below, however, for the purposes of tracking periodic orbits this number of degrees of freedom is too high. In the next section, we will introduce a reduction by symmetry that makes the problem tractable.

In order to observe the inertial range spectrum over one decade we need to have $R_\lambda \gtrsim 100$ and $N = 256$ is the minimal resolution. This truncation level has about $1.5 \cdot 10^7$ independent Fourier modes. For time being, this number is too large to handle even after reduction by symmetry.

3.2. Reduction by symmetry

In order to reduce the number of degrees of freedom in our simulations we impose spatial symmetries on the solutions. The set of symmetries we impose is a subgroup of the full symmetry group of the boundary-free Navier–Stokes equation, which consists of translations, rotations and reflections. The maximal set of symmetries that allows for turbulent solutions was determined by Kida in Ref. 13 and reduces the number of degrees of freedom by a factor of nearly 200. The resulting, so called high-symmetric, flow can be studied at high microscale Reynolds number with a relatively small computational effort. In the eighties and early nineties, this enabled Kida et al.\textsuperscript{12} to observe the Kolmogorov scaling laws in numerical experiment (arguably) for the very first time. With present-day computers, however, these scaling laws can be reproduced without any symmetry constraints.
The equations for high-symmetric flow found a new use in the search for finite-time blowup solutions to the Navier–Stokes equations, and are in that context referred to as Kida-Pelz flow.

As will be explained below, the method we use for finding and continuing periodic orbits requires the computation and decomposition of matrices of the size of the number of degrees of freedom. Therefore, the reduction by symmetry from $2 \cdot 10^5$ down to $1 \cdot 10^4$ degrees of freedom is essential. Thus, we recycled an old idea for reduction of the size of the problem in order to test and explore a new idea for the analysis of the problem.

3.3. How to find unstable solutions?

After reduction by symmetry we have a system of $n \approx 10,000$ coupled, nonlinear ordinary differential equations with one parameter, $\nu$. Symbolically, we can write the dynamical equations as

$$\frac{d}{dt} x = f(x, \nu)$$

where $x$ is the $n$-dimensional vector that holds the linearly independent Fourier components of vorticity. In this system of equations we want to find periodic solutions. It is convenient to regard them as fixed points of an iterated Poincaré map in phase space. We fix a plane of intersection $S$ by setting one of the small wave number components of vorticity to a constant, e.g. $x_1 = c$. The coordinates $y$ in the intersection plane are the remaining $(n - 1)$ components. Fixed points of the Poincaré map $P$ on $S$ satisfy

$$P^m(y) - y = 0 \quad (m = 1, 2, 3, \cdots),$$

for some “discrete period” $m$. The Poincaré map is found by integrating Eq. 3 over a finite time interval and consequently Eq. 4 is highly nonlinear. In order to find solutions numerically we use Newton–Raphson iteration. Essential for the convergence of this method is an accurate initial guess.

As explained in the introduction, we do not extract initial data from the bifurcation diagram at low microscale Reynolds number. Instead, we filter initial guesses for the Newton-Raphson iterations directly from a long, weakly turbulent time series. We run a simulation at $R_\lambda \approx 55$ and keep track of the iterates of the Poincaré map. If a point is mapped close to itself, i.e.

$$\|P^m(y) - y\|_Q < \delta,$$

we use it as an initial guess. The distance is measured as the enstrophy of the difference field and $\delta$ is a threshold that we found by trial and error.
to be around 10% of the standard deviation of enstrophy at this $R_\lambda$. Less accurate initial guesses usually diverge under Newton–Raphson iteration. This is essentially the same procedure as the one used in Ref. 10.

In order to perform Newton-Raphson iteration on the initial guesses we need to compute the matrix of derivatives of the Poincaré map. For this end we used finite differencing, which means that we have to integrate Eq. 3 over a finite time interval once for each degree of freedom, which can be done efficiently on parallel processors. This is computationally the hardest step, and the time and money spent on this part of the process is certainly the largest obstacle we have to overcome to study periodic orbits in turbulence.

In the case at hand, we filtered about 50 initial guesses from the time series, from which we distilled a dozen or so periodic orbits. The discrete period of these orbits is closely related to their period in the continuous-time system. An orbit with a discrete period $m$ has a period roughly equal to $mT_R$, where $T_R$ is the most probable return time of the Poincaré map. Thus we can refer to the periodic orbits we found as period-$m$ orbits.

3.4. Continuation in the Reynolds number

Our primary interest is to see if the periodic orbits can represent the onset of developed turbulence. Therefore, we continued orbits with period 1 up to 5 in $\nu$ and compared the time-mean energy dissipation rate to that of turbulence. The result is shown in Fig. 2. Interestingly, the time-mean values produced by the periodic orbits are all close to that of turbulence at $\nu = 0.0045$ ($R_\lambda = 55$), where they have been filtered from a time series. If we decrease $\nu$ only the orbit of longest period reproduces the turbulent values accurately. The period of this orbit is about $2.5T_T$, where $T_T$ is the large-eddy turnover time, i.e. the longest intrinsic time scale of turbulence.

This observation leads us to the conjecture that the period-5 orbit represents the onset of developed turbulence, in other words it is embedded in the turbulent attractor in the whole range of $R_\lambda$. In Ref. 1 it is shown that also the energy spectrum agrees extremely well with that of turbulence - both simulations, experiments and theory.

3.5. Analysis of embedded periodic motion

By comparing time-mean quantities we have established that the period-5 orbits represents the turbulent motion. Now we want to learn about the turbulent dynamics by studying the time-periodic solution. One interesting quantity that is hard, if not impossible, to compute for turbulence is the
Fig. 2. Continuation of the periodic orbits in the viscosity. Shown is the time-mean energy dissipation rate for turbulence (blue), period-1 to period-4 (green) and period-5 (red). The period-5 orbit reproduces the values measured in turbulence well, especially in the regime of developed turbulence, i.e. $\nu < 0.004 \ (R_\lambda > 60)$.

Lyapunov spectrum. The largest exponent can readily be computed, but exponents deeper in the spectrum are swamped by numerical error in practical computations. Consequently, little is known about the properties of the Lyapunov spectrum of turbulence and its relation to physical processes. Here, we use the period-5 orbit as a reference solution to obtain a number of time-mean and local exponents. These data indicate that the Lyapunov spectrum is related to the energy spectrum through preferred spatial scales of growing and decaying perturbations.

The computation of the Lyapunov spectrum requires integration of the linearised equations

$$\frac{d}{dt} \mathbf{v} = J \mathbf{v},$$

where $J$ is the matrix of derivatives of the vector field $\mathbf{f}$ in Eq. 3 and $\mathbf{v}$ is a perturbation vorticity field. Through $J$, the time evolution of the perturbation field depends on the reference orbit $\mathbf{x}(t)$. The Lyapunov exponents are defined by

$$\Lambda = \lim_{t \to \infty} \frac{1}{2t} \ln \frac{\| \mathbf{v}(t) \|_Q}{\| \mathbf{v}(0) \|_Q}$$

(7)
The Oseledec theorem, also known as the multiplicative ergodic theorem, ensures that for each initial point \( x(0) \) there exist \( n \) independent initial perturbations \( v_i(0) \) that give rise to distinct exponents \( \Lambda_1 > \Lambda_2 > \ldots > \Lambda_n \). The spectrum of exponents does not depend on the initial point, as long as it lies on a chaotic orbit which samples the whole attractor. \(^a\) The computation of more than one exponent is necessary for estimating such quantities central to chaos theory as the attractor dimension and the Kolmogorov-Sinai entropy.\(^{14}\)

In addition to the time-mean exponents we can compute the local Lyapunov exponents, corresponding to the instantaneous growth rate of the perturbation fields. The local exponents are defined as

\[
\lambda_i(t) = \frac{1}{2} \frac{d}{dt} \ln \| v_i(t) \|_Q
\]

so that their time-mean value coincides with the Lyapunov spectrum, i.e., \( \bar{\lambda}_i = \Lambda_i \). A very long time integration is necessary for the computation of the spectrum as the local exponents typically have a standard deviation an order or magnitude larger than their mean value. In order to compute the largest \( k \) exponents we need to integrate Eqs. 3 and 6 for \( k \) independent initial perturbations. At regular time intervals during the integration an orthogonalisation procedure is executed to separate the distinct growth rates. As some of the perturbations are exponentially growing while others are exponentially decaying, the orthogonalisation introduces a significant error which accumulates. Consequently, accurate estimates are hard to obtain, and in Ref. 9 the authors take care to state their results as approximate lower bounds on the attractor dimension.

If we compute the Lyapunov exponents relative to a periodic orbit we can apply Floquet theory to Eq. 6. It then follows that the exponents \( \Lambda_i \) are directly related to the eigenvalues of the monodromy matrix, which, after a projection onto the plane of intersection, is equal to the matrix of derivatives of the Poincaré map. The eigenvectors of the monodromy matrix correspond to the initial perturbations \( v_i(0) \), also called Lyapunov vectors. Therefore, once we have computed the eigenspectrum of the monodromy matrix we can compute any number of time-mean Lyapunov exponents and a large number of local Lyapunov exponents. The number of local exponents we can compute is only restricted by the accumulation of numerical error in the integration of Eqs. 3 and 6 over one period.

\(^a\)Technically, the spectrum is equal for almost all initial points relative to the natural invariant measure on the attractor. It differs if the initial point lies on a periodic orbit.
We computed the leading 50 local Lyapunov exponents along the period-5 orbit at $R_\lambda = 67$. The largest exponent is close to that of turbulent motion. In units $T^{-1}$, we find that $A_1 = 0.88$ and $A_1^{5p} = 1.05$ for the turbulent and the period-5 motion, respectively, compared to a standard deviation of the corresponding local exponent in turbulence of 1.74. In Fig. 3 we visualise the behaviour of the local exponents over one period, and compare it to the instantaneous energy spectrum. In each subplot, we include the time series of the energy input rate on the left and that of the energy dissipation rate on the right. The way to read this figure is as follows: energy is input at small wave numbers, and therefore the energy content at small wave numbers is strongly correlated to the energy input rate. In agreement with the conventional picture of the energy cascade process, we see the local maxima progress to larger wave numbers in the course of time. The energy content in the dissipation range is strongly correlated to the energy dissipation rate. In the second subplot we can see a similar correlation: the local exponents with small indices are correlated to the energy input rate, whereas those with a large index are correlated to the energy dissipation rate. This is intuitively correct, as we tend to associate instabilities with external forcing and damped perturbations with dissipation. However, apart from some speculations arising from shell model turbulence\cite{15}, to our knowledge no relation had ever been established between the Lyapunov spectrum of turbulence on one hand and spatial scales and physical processes on the other.

We expect to see an even clearer correlation between the Lyapunov spectrum and the energy cascade process in turbulence with a developed inertial range. Ultimately, we should be able to establish a direct relation between the processes in physical space which contribute to the energy cascade, and the mathematical properties of the attractor in phase space.

4. Conclusions and Outlook

The work on isotropic turbulence is ongoing. We are studying the spatial patterns represented by, in particular, the period-5 solution. Also, we are exploring ways to increase the truncation level and study periodic orbits in the presence of an inertial range. Thirdly, an interesting issue is the role played by the period of the orbits. We are gathering orbits of longer period for intercomparison to shine a light on this issue.

Obviously the work presented here poses more questions than it answers. It is only a first glance at the results that can be obtained by applying
dynamical systems theory to turbulent flows. Moreover, this first glance is blurred by the practical restrictions we face when performing the numerical computations. We had to impose spatial symmetries, which renders the resulting flow non-homogeneous. We had to choose a fairly low resolution. We had to restrict ourselves to the analysis of five orbits of fairly short period. Yet we obtained promising results, and similarly rapid progress is made in the study of shear flows and the development of efficient numerical methods.\textsuperscript{2,3,16,17} This fresh attack on one of the oldest standing problems of physics will, at the very least, bring the theory of dynamical systems and the theory of turbulence closer together and be an inspiration to both.

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Fig. 3. Top: the instantaneous (3D) energy spectrum along the period-5 orbit, with on the left the energy input rate and on the right the energy dissipation rate. Local maxima of \( e(t) \) give rise to local maxima in the energy content at small wave number, which cascade to larger wave numbers and show correlation to \( e(t) \) with a small time delay in the dissipation range. Bottom: time series of the local Lyapunov exponents visualised with contours just like for the energy spectrum. There is a positive correlation between the local Lyapunov exponents with a small index (\( \Lambda_1 \) to \( \Lambda_{19} \)) and \( e(t) \) and between those with large index (\( \Lambda_{20} \) and higher) and \( e(t) \). Incidentally, the Kaplan-Yorke dimension computed from the \( \Lambda_i \) is 19.7. Contours denote deviation from time mean, normalised by standard deviation. Red denotes positive and blue negative deviations.